Discrete Mathematics in Computer Science

B7. Sets: Countability

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B7.1 Countable Sets

Comparing Cardinality

- Two sets A and B have the same cardinality if their elements can be paired (i.e. there is a bijection from A to B).
- Set A has a strictly smaller cardinality than set B if
  - we can map distinct elements of A to distinct elements of B (i.e. there is an injective function from A to B), and
  - $|A| \neq |B|$.
- This clearly makes sense for finite sets.
- What about infinite sets? Do they even have different cardinalities?

Countable and Countably Infinite Sets

Definition (countably infinite and countable)
A set A is countably infinite if $|A| = |\mathbb{N}_0|$.

A set A is countable if $|A| \leq |\mathbb{N}_0|$.

A set is countable if it is finite or countably infinite.

- We can count the elements of a countable set one at a time.
- The objects are "discrete" (in contrast to "continuous").
- Discrete mathematics deals with all kinds of countable sets.

Set of Even Numbers

- $\mathit{even} = \{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}$
- Obviously: $\mathit{even} \subset \mathbb{N}_0$
- Intuitively, there are twice as many natural numbers as even numbers — no?
- Is $|\mathit{even}| < |\mathbb{N}_0|$?

Set of Even Numbers

Theorem (set of even numbers is countably infinite)
The set of all even natural numbers is countably infinite,
i. e. $|\{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}| = |\mathbb{N}_0|$.

Proof Sketch.
We can pair every even number $2n$ with natural number n. □

Set of Perfect Squares

Theorem (set of perfect squares is countably infinite)
The set of all perfect squares is countably infinite,
i. e. $|\{n^2 \mid n \in \mathbb{N}_0\}| = |\mathbb{N}_0|$.

Proof Sketch.
We can pair every square number n^2 with natural number n. □

Subsets of Countable Sets are Countable

In general:

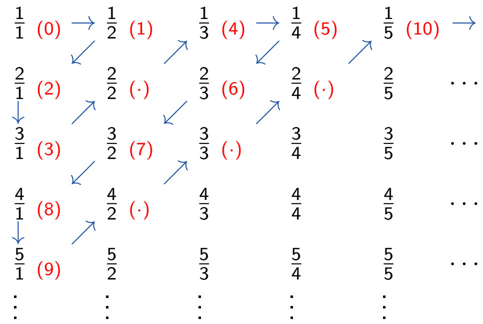
Theorem (subsets of countable sets are countable)
Let A be a countable set. Every set B with $B \subseteq A$ is countable.

Proof.
Since A is countable there is an injective function f from A to $\mathbb{N}_0$.
The restriction of f to B is an injective function from B to $\mathbb{N}_0$. □

Set of the Positive Rationals

Theorem (set of positive rationals is countably infinite)
Set $\mathbb{Q}_+ = \{n \mid n \in \mathbb{Q} \text{ and } n > 0\} = \{p/q \mid p, q \in \mathbb{N}_1\}$ is countably infinite.

Proof idea.



□

Union of Two Countable Sets is Countable

Theorem (union of two countable sets countable)
Let A and B be countable sets. Then $A \cup B$ is countable.

Proof sketch.
As A and B are countable there is an injective function f_A from A to $\mathbb{N}_0$, analogously f_B from B to $\mathbb{N}_0$.

We define function $f_{A \cup B}$ from $A \cup B$ to $\mathbb{N}_0$ as

$$f_{A \cup B}(e) = \begin{cases} 2f_A(e) & \text{if } e \in A \\ 2f_B(e) + 1 & \text{otherwise} \end{cases}$$

This $f_{A \cup B}$ is an injective function from $A \cup B$ to $\mathbb{N}_0$. □

Integers and Rationals

Theorem (sets of integers and rationals are countably infinite)
The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.

Without proof ($\rightsquigarrow$ exercises)

Union of More than Two Sets

Definition (arbitrary unions)

Let M be a set of sets. The union $\bigcup_{S \in M} S$ is the set with

$$x \in \bigcup_{S \in M} S \text{ iff exists } S \in M \text{ with } x \in S.$$

Countable Union of Countable Sets

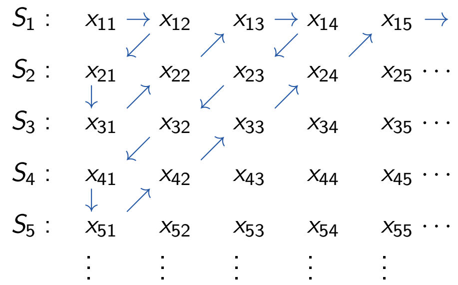
Theorem

Let M be a countable set of countable sets.

Then $\bigcup_{S \in M} S$ is countable.

Proof sketch.

With $M = \{S_1, S_2, S_3, \dots\}$ (possibly finite) and each $S_i = \{x_{i1}, x_{i2}, \dots\}$ (possibly finite), we can use an analogous idea as for the countability of $\mathbb{Q}_+$ (skipping duplicates):



□

Set of all Binary Trees is Countable

Theorem (set of all binary trees is countable)

The set $B = \{b \mid b \text{ is a binary tree}\}$ is countable.

Proof.

For $n \in \mathbb{N}_0$ the set B_n of all binary trees with n leaves is finite.

With $M = \{B_i \mid i \in \mathbb{N}_0\}$ the set of all binary trees is $B = \bigcup_{B' \in M} B'$.

Since M is a countable set of countable sets, B is countable. □

And Now?

We have seen several countably infinite sets.

What about our original questions?

- Do all infinite sets have the same cardinality?
- Are they all countably infinite?

Summary

- A set is countable if it has at most cardinality $|\mathbb{N}_0|$.
- If a set is countable and infinite, it is countably infinite.
- Sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.
- Every subset of a countable set is countable.
- Every countable union of countable sets is countable, in particular, the union of two countable sets is countable.