Discrete Mathematics in Computer Science
B7. Sets: Countability

Malte Helmert, Gabriele Röger

University of Basel

October 28, 2024

B7.1 Countable Sets

B7.1 Countable Sets

Comparing Cardinality

- ▶ Two sets A and B have the same cardinality
  if their elements can be paired
  (i.e. there is a bijection from A to B).
- ▶ Set A has a strictly smaller cardinality than set B if
  - ▶ we can map distinct elements of A to distinct elements of B
    (i.e. there is an injective function from A to B), and
  - ▶ $|A| \neq |B|$.

- ▶ This clearly makes sense for finite sets.
- ▶ What about infinite sets?
  Do they even have different cardinalities?

Countable and Countably Infinite Sets

Definition (countably infinite and countable)

A set A is countably infinite if $|A| = |\mathbb{N}_0|$.

A set A is countable if $|A| \leq |\mathbb{N}_0|$.

A set is countable if it is finite or countably infinite.

▶ We can count the elements of a countable set one at a time.

▶ The objects are "discrete" (in contrast to "continuous").

▶ Discrete mathematics deals with all kinds of countable sets.

Set of Even Numbers

- ▶ $even = \{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}$
- ▶ Obviously: $even \subset \mathbb{N}_0$
- ▶ Intuitively, there are twice as many natural numbers as even numbers — no?
- ▶ Is $|even| < |\mathbb{N}_0|$?

Set of Even Numbers

Theorem (set of even numbers is countably infinite)

The set of all red *even natural numbers* *is* red *countably infinite*,
i. e. $|\{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}| = |\mathbb{N}_0|$.

Proof Sketch.
We can pair every even number $2n$ with natural number n. □

Set of Perfect Squares

> **Theorem (set of perfect squares is countably infinite)**
> *The set of all perfect squares is countably infinite,*
> *i. e.* $|\{n^2 \mid n \in \mathbb{N}_0\}| = |\mathbb{N}_0|$.

> **Proof Sketch.**
> We can pair every square number n^2 with natural number n. □

Subsets of Countable Sets are Countable

In general:

Theorem (subsets of countable sets are countable)
Let A be a countable set. Every set B with $B \subseteq A$ is countable.
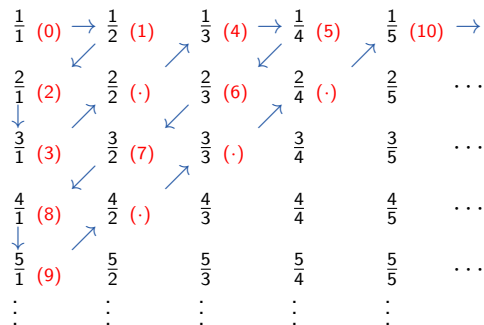
Proof.
Since A is countable there is an injective function f from A to $\mathbb{N}_0$.
The restriction of f to B is an injective function from B to $\mathbb{N}_0$. □

Set of the Positive Rationals

Theorem (set of positive rationals is countably infinite)

Set $\mathbb{Q}_+ = \{n \mid n \in \mathbb{Q} \text{ and } n > 0\} = \{p/q \mid p, q \in \mathbb{N}_1\}$
is countably infinite.

Proof idea.

$\frac{1}{1}$ (0) $\to$ $\frac{1}{2}$ (1) $\frac{1}{3}$ (4) $\to$ $\frac{1}{4}$ (5) $\frac{1}{5}$ (10) $\to$

$\frac{2}{1}$ (2) $\frac{2}{2}$ ($\cdot$) $\frac{2}{3}$ (6) $\frac{2}{4}$ ($\cdot$) $\frac{2}{5}$ $\cdots$

$\frac{3}{1}$ (3) $\frac{3}{2}$ (7) $\frac{3}{3}$ ($\cdot$) $\frac{3}{4}$ $\frac{3}{5}$ $\cdots$

$\frac{4}{1}$ (8) $\frac{4}{2}$ ($\cdot$) $\frac{4}{3}$ $\frac{4}{4}$ $\frac{4}{5}$ $\cdots$

$\frac{5}{1}$ (9) $\frac{5}{2}$ $\frac{5}{3}$ $\frac{5}{4}$ $\frac{5}{5}$ $\cdots$

$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\square$

Union of Two Countable Sets is Countable

Theorem (union of two countable sets countable)
Let A and B be countable sets. Then $A \cup B$ is countable.

Proof sketch.
As A and B are countable there is an injective function f_A from A
to $\mathbb{N}_0$, analogously f_B from B to $\mathbb{N}_0$.

We define function $f_{A \cup B}$ from $A \cup B$ to $\mathbb{N}_0$ as

$$f_{A \cup B}(e) = \begin{cases} 2f_A(e) & \text{if } e \in A \\ 2f_B(e) + 1 & \text{otherwise} \end{cases}$$

This $f_{A \cup B}$ is an injective function from $A \cup B$ to $\mathbb{N}_0$. □

Integers and Rationals

> **Theorem (sets of integers and rationals are countably infinite)**
> *The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.*

Without proof ($\rightsquigarrow$ exercises)

Union of More than Two Sets

Definition (arbitrary unions)

Let M be a set of sets. The union $\bigcup_{S \in M} S$ is the set with

$$x \in \bigcup_{S \in M} S \text{ iff exists } S \in M \text{ with } x \in S.$$
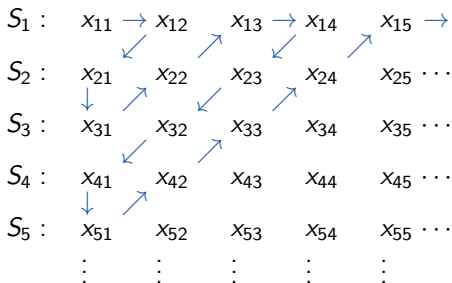
Countable Union of Countable Sets

Theorem
Let M be a countable set of countable sets.

Then $\bigcup_{S \in M} S$ is countable.

Proof sketch.
With $M = \{S_1, S_2, S_3, \dots\}$ (possibly finite) and each
$S_i = \{x_{i1}, x_{i2}, \dots\}$ (possibly finite), we can use an analogous idea
as for the countability of $\mathbb{Q}_+$ (skipping duplicates):

$$
\begin{array}{llllll}
S_1: & x_{11} \rightarrow x_{12} & x_{13} \rightarrow x_{14} & x_{15} \rightarrow \\
S_2: & x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \cdots \\
S_3: & x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \cdots \\
S_4: & x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \cdots \\
S_5: & x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \cdots \\
\end{array}
$$

Set of all Binary Trees is Countable

> **Theorem (set of all binary trees is countable)**
>
> *The set $B = \{b \mid b \text{ is a binary tree}\}$ is countable.*

> **Proof.**
>
> For $n \in \mathbb{N}_0$ the set B_n of all binary trees with n leaves is finite.
>
> With $M = \{B_i \mid i \in \mathbb{N}_0\}$ the set of all binary trees is
> $B = \bigcup_{B' \in M} B'$.
>
> Since M is a countable set of countable sets, B is countable.   □

And Now?

We have seen several countably infinite sets.

What about our original questions?

- ▶ Do all infinite sets have the same cardinality?
- ▶ Are they all countably infinite?

Summary

- ▶ A set is countable if it has at most cardinality $|\mathbb{N}_0|$.
- ▶ If a set is countable and infinite, it is countably infinite.
- ▶ Sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.
- ▶ Every subset of a countable set is countable.
- ▶ Every countable union of countable sets is countable, in particular, the union of two countable sets is countable.