# Discrete Mathematics in Computer Science

## B6. Sets: Comparing Cardinality and Hilbert's Hotel

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# B6.1 Comparing Cardinality





## Finite Sets Revisited

We already know:

- The cardinality $|S|$ measures the size of set $S$.
- A set is finite if it has a finite number of elements.
- The cardinality of a finite set is the number of elements it contains.

A set is infinite if it has an infinite number of elements.

Do all infinite sets have the same cardinality?

## Comparing the Cardinality of Sets

- Consider $A = \{1, 2\}$ and $B = \{\text{dog}, \text{cat}, \text{mouse}\}$.
- We can map distinct elements of $A$ to distinct elements of $B$, e.g.

$$1 \mapsto \text{dog}$$
$$2 \mapsto \text{cat}$$

- This is an injective function from $A$ to $B$:
  - every element of $A$ is mapped to an element of $B$;
  - different elements of $A$ are mapped to different elements of $B$.





## Comparing Cardinality

**Definition (cardinality not larger)**

Set $A$ has cardinality less than or equal to the cardinality of set $B$ ($|A| \leq |B|$), if there is an injective function from $A$ to $B$.





## Comparing the Cardinality of Sets

- $A = \{1, 2, 3\}$ and $B = \{\text{dog}, \text{cat}, \text{mouse}\}$ have cardinality 3.
- We can pair their elements by a bijection from $A$ to $B$:

$$1 \leftrightarrow \text{dog}$$
$$2 \leftrightarrow \text{cat}$$
$$3 \leftrightarrow \text{mouse}$$

- This is a bijection from $A$ to $B$.
  - Each element of $A$ is paired with exactly one element of set $B$.
  - Each element of $B$ is paired with exactly one element of $A$.
- If there is a bijection from $A$ to $B$ there is one from $B$ to $A$ (the inverse function).





## Equinumerous Sets

We use the existence of a bijection also as criterion for infinite sets:

**Definition (equinumerous sets)**

Two sets $A$ and $B$ have the same cardinality ($|A| = |B|$) if there exists a bijection from $A$ to $B$.

Such sets are called equinumerous.

**Definition (strictly smaller cardinality)**

Set $A$ has cardinality strictly less than the cardinality of set $B$ ($|A| < |B|$), if $|A| \leq |B|$ and $|A| \neq |B|$.

Consider set $A$ and object $e \notin A$. Is $|A| < |A \cup \{e\}|$?

# B6.2 Hilbert's Hotel

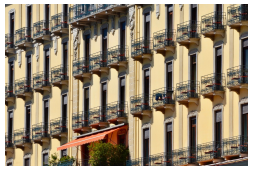




## Hilbert's Hotel

Our intuition for finite sets does not always work for infinite sets.

- If in a hotel all rooms are occupied then it cannot accomodate additional guests.
- But Hilbert's Grand Hotel has infinitely many rooms.
- All these rooms are occupied.







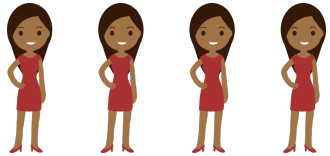
## One More Guest Arrives

- Every guest moves from her current room $n$ to room $n + 1$.
- Room 1 is then free.
- The new guest gets room 1.





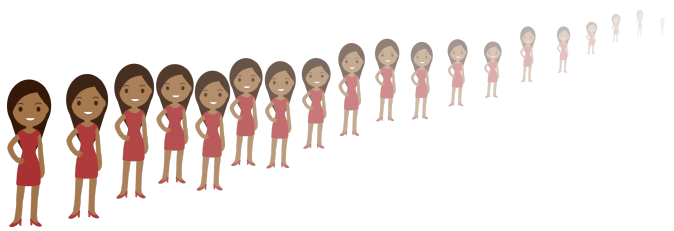
## Four More Guests Arrive



- Every guest moves from her current room $n$ to room $n + 4$.
- Rooms 1 to 4 are no longer occupied and can be used for the new guests.

$\rightarrow$ Works for any finite number of additional guests.

## An Infinite Number of Guests Arrives



- Every guest moves from her current room $n$ to room $2n$.
- The infinitely many rooms with odd numbers are now available.
- The new guests fit into these rooms.





## Can we Go further?

What if ...

- infinitely many coaches, each with an infinite number of guests
- infinitely many ferries, each with an infinite number of coaches, each with infinitely many guests
- ...

... arrive?

There are strategies for all these situations
as long as with "infinite" we mean "countably infinite"
and there is a finite number of layers.





## Summary

- Set $A$ has cardinality less than or equal the cardinality of set $B$ ($|A| \leq |B|$), if there is an injective function from $A$ to $B$.
- Sets A and B have the same cardinality ($|A| = |B|$) if there exists a bijection from A to B.
- Our intuition for finite sets does not always work for infinite sets.