Discrete Mathematics in Computer Science

B6. Sets: Comparing Cardinality and Hilbert's Hotel

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B6.1 Comparing Cardinality

Finite Sets Revisited

We already know:

- The cardinality $|S|$ measures the size of set S.
- A set is finite if it has a finite number of elements.
- The cardinality of a finite set
  is the number of elements it contains.

A set is infinite if it has an infinite number of elements.

Do all infinite sets have the same cardinality?

Comparing the Cardinality of Sets

- Consider $A = \{1, 2\}$ and $B = \{\text{dog}, \text{cat}, \text{mouse}\}$.
- We can map distinct elements of A to distinct elements of B, e.g.

$$1 \mapsto \text{dog}$$
$$2 \mapsto \text{cat}$$

- This is an injective function from A to B:
  - every element of A is mapped to an element of B;
  - different elements of A are mapped to different elements of B.

Comparing Cardinality

Definition (cardinality not larger)

Set A has cardinality less than or equal to the cardinality of set B ($|A| \leq |B|$), if there is an injective function from A to B.

Comparing the Cardinality of Sets

- $A = \{1, 2, 3\}$ and $B = \{\text{dog}, \text{cat}, \text{mouse}\}$ have cardinality 3.
- We can pair their elements by a bijection from A to B:

$$1 \leftrightarrow \text{dog}$$
$$2 \leftrightarrow \text{cat}$$
$$3 \leftrightarrow \text{mouse}$$

- This is a bijection from A to B.
  - Each element of A is paired with exactly one element of set B.
  - Each element of B is paired with exactly one element of A.
- If there is a bijection from A to B there is one from B to A (the inverse function).

Equinumerous Sets

We use the existence of a bijection also as criterion for infinite sets:

Definition (equinumerous sets)

Two sets A and B have the same cardinality ($|A| = |B|$) if there exists a bijection from A to B.

Such sets are called equinumerous.

Definition (strictly smaller cardinality)

Set A has cardinality strictly less than the cardinality of set B ($|A| < |B|$), if $|A| \leq |B|$ and $|A| \neq |B|$.

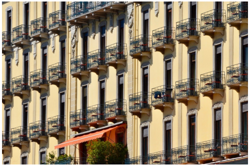
Consider set A and object $e \notin A$. Is $|A| < |A \cup \{e\}|$?

B6.2 Hilbert's Hotel

Hilbert's Hotel

Our intuition for finite sets does not always work for infinite sets.

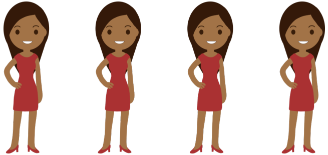
- If in a hotel all rooms are occupied then it cannot accomodate additional guests.
- But Hilbert's Grand Hotel has infinitely many rooms.
- All these rooms are occupied.



One More Guest Arrives

- Every guest moves from her current room n to room $n + 1$.
- Room 1 is then free.
- The new guest gets room 1.

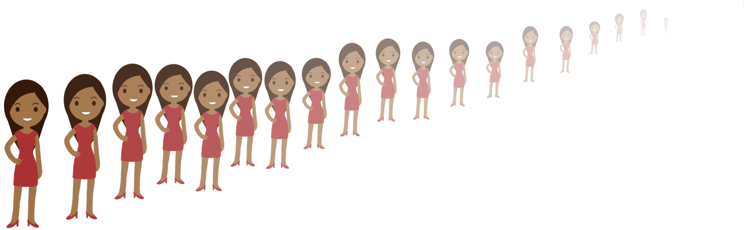
Four More Guests Arrive



- Every guest moves from her current room n to room $n + 4$.
- Rooms 1 to 4 are no longer occupied and can be used for the new guests.

$\rightarrow$ Works for any finite number of additional guests.

An Infinite Number of Guests Arrives



- Every guest moves from her current room n to room $2n$.
- The infinitely many rooms with odd numbers are now available.
- The new guests fit into these rooms.

Can we Go further?

What if . . .

- infinitely many coaches, each with an infinite number of guests
- infinitely many ferries, each with an infinite number of coaches, each with infinitely many guests
- . . .

. . . arrive?

There are strategies for all these situations
as long as with "infinite" we mean "countably infinite"
and there is a finite number of layers.

Summary

- Set A has cardinality less than or equal the cardinality of set B ($|A| \leq |B|$), if there is an injective function from A to B.
- Sets A and B have the same cardinality ($|A| = |B|$) if there exists a bijection from A to B.
- Our intuition for finite sets does not always work for infinite sets.