Discrete Mathematics in Computer Science

B4. Operations on Relations

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Operations on Relations

Relations: Recap

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- A homogeneous relation R over set S is a binary relation $R \subseteq S \times S$.

Set Operations

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  With the standard relations for $\mathbb{N}_0$, relation $=$ is the complementary relation of $\neq$ and $>$ the one of $\leq$.

Inverse of a Relation

Definition

Let $R \subseteq A \times B$ be a binary relation over A and B.

The inverse relation of R is the relation $R^{-1} \subseteq B \times A$ given by $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

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- The inverse of the $<$ relation over $\mathbb{N}_0$ is the $>$ relation.
- Relation R with xRy iff person x has a key for y.
  Inverse: Q with aQb iff lock a can be opened by person b.

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Composition of Relations

Definition (Composition of relations)

Let R_1 be a relation over A and B and R_2 a relation over B and C.

The composition of R_1 and R_2 is the relation $R_2 \circ R_1$ over A and C with:

$$R_2 \circ R_1 = \{(a, c) \mid \textit{there is a } b \in B \text{ with } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

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How can we illustrate this graphically?

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Composition of Relations: Example

$S_1 = \{1, 2, 3, 4\}$
$S_2 = \{A, B, C, D, E\}$
$S_3 = \{a, b, c, d\}$

$R_1 = \{(1, A), (1, B), (3, B), (4, D)\}$ over S_1 and S_2

$R_2 = \{(B, a), (C, c), (D, a), (D, d)\}$ over S_2 and S_3

$R_2 \circ R_1 =$

Composition is Associative

Theorem (Associativity of composition)

Let $S_1, \ldots, S_4$ be sets and R_1, R_2, R_3 relations with $R_i \subseteq S_i \times S_{i+1}$. Then

$$R_3 \circ (R_2 \circ R_1) = (R_3 \circ R_2) \circ R_1.$$

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Proof.

It holds that $(x_1, x_4) \in R_3 \circ (R_2 \circ R_1)$ iff there is an x_3 with $(x_1, x_3) \in R_2 \circ R_1$ and $(x_3, x_4) \in R_3$.

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As $(x_1, x_3) \in R_2 \circ R_1$ iff there is an x_2 with $(x_1, x_2) \in R_1$ and $(x_2, x_3) \in R_2$, we have overall that $(x_1, x_4) \in R_3 \circ (R_2 \circ R_1)$ iff there are x_2, x_3 with $(x_1, x_2) \in R_1$, $(x_2, x_3) \in R_2$ and $(x_3, x_4) \in R_3$.

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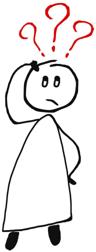
This is the case iff there is an x_2 with $(x_1, x_2) \in R_1$ and $(x_2, x_4) \in R_3 \circ R_2$, which holds iff $(x_1, x_4) \in (R_3 \circ R_2) \circ R_1$. □

Questions



Questions?

(Reflexive) Transitive Closure

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Let R be a relation over set S.

The transitive closure R^+ of R is the smallest relation over S that is transitive and has R as a subset.

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Example: If aRb specifies that there is a direct flight from a to b, what do R^+ and R^* express?

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Transitive Closure and n-fold Composition

Define the *n-fold composition* of a relation R over S as

$$R_0 = \{(x, x) \mid x \in S\} \qquad \text{and}$$
$$R_i = R \circ R_{i-1} \qquad \text{for } i > 1.$$

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$$R_i = R \circ R_{i-1} \qquad \text{for } i > 1.$$

Theorem

Let R be a relation over set S.
Then $R^+ = \bigcup_{i=1}^{\infty} R_i$ and $R^* = \bigcup_{i=0}^{\infty} R_i$.

Without proof.

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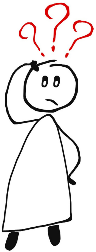

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- Highly relevant for queries over relational databases.
- For example, join operators combine relations based on common entries.
- Example for a natural join:

Employee

Name	EmpId	DeptName
Harry	3415	Finance
Sally	2241	Sales
George	3401	Finance
Harriet	2202	Sales
Mary	1257	Human Resources

Dept

DeptName	Manager
Finance	George
Sales	Harriet
Production	Charles

Employee ⋈ Dept

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Sally	2241	Sales	Harriet
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(Source: Wikipedia)

Summary

- Relations: general, binary, homogeneous
- Properties: reflexivity, symmetry, transitivity
  (and related properties)
- Special relations: equivalence relations, order relations
- Operations: inverse, composition, transitive closure