

Discrete Mathematics in Computer Science

B3. Equivalence and Order Relations

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B3.1 Equivalence Relations

B3.2 Order Relations

B3.1 Equivalence Relations

Motivation

- ▶ Think of any attribute that two objects can have in common, e. g. their color.
- ▶ We could place the objects into distinct “buckets”, e. g. one bucket for each color.
- ▶ We also can define a relation \sim such that $x \sim y$ iff x and y share the attribute, e. g. have the same color.
- ▶ Would this relation be
 - ▶ reflexive?
 - ▶ irreflexive?
 - ▶ symmetric?
 - ▶ asymmetric?
 - ▶ antisymmetric?
 - ▶ transitive?

Equivalence Relation

Definition (Equivalence Relation)

A binary relation \sim over set S is an **equivalence relation** if \sim is **reflexive, symmetric and transitive**.

Examples:

- ▶ $\{(x, y) \mid x \text{ and } y \text{ have the same place of origin}\}$
over the set of all Swiss citizens
- ▶ $\{(x, y) \mid x \text{ and } y \text{ have the same parity}\}$ over \mathbb{N}_0
- ▶ $\{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), (2, 2), (2, 3), (3, 2), (3, 3)\}$ over $\{1, 2, \dots, 5\}$

Is this definition indeed what we want?

Does it allow us to partition the objects into buckets

(e. g. one “bucket” for all objects that share a specific color)?

German: Äquivalenzrelation

Equivalence Classes

Definition (Equivalence Class)

Let \sim be an equivalence relation over set S .

For any $x \in S$, the **equivalence class of x** is the set

$$[x]_{\sim} = \{y \in S \mid x \sim y\}.$$

Consider

$$\sim = \{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), \\ (2, 2), (2, 3), (3, 2), (3, 3)\}$$

over set $\{1, 2, \dots, 5\}$.

$$[4]_{\sim} =$$

German: Äquivalenzklasse

Equivalence Classes: Properties

Let \sim be an equivalence relation over set S and $E = \{[x]_{\sim} \mid x \in S\}$ the set of all equivalence classes.

- ▶ Every element of S is in some equivalence class in E .
- ▶ Every element of S is in at most one equivalence class in E .
 \rightsquigarrow homework assignment

\Rightarrow Equivalence relations induce **partitions**
(not covered in this course).

B3.2 Order Relations

Order Relations

- ▶ We now consider **other combinations of properties**, that allow us to describe a **consistent order** of the objects.
- ▶ “Number x is not larger than number y .”
“Set S is a subset of set T .”
“Jerry runs at least as fast as Tom.”
“Pasta tastes better than Potatoes.”

German: Ordnungsrelation

Partial Orders

- ▶ We begin with **partial orders**.
- ▶ Example partial order relations are \leq over \mathbb{N}_0 or \subseteq for sets.
- ▶ Are these relations
 - ▶ reflexive?
 - ▶ irreflexive?
 - ▶ symmetric?
 - ▶ asymmetric?
 - ▶ antisymmetric?
 - ▶ transitive?

Partial Orders – Definition

Definition (Partial order)

A binary relation \preceq over set S is a **partial order** if \preceq is **reflexive, antisymmetric and transitive**.

Which of these relations are partial orders?

- ▶ strict subset relation \subset for sets
- ▶ not-less-than relation \geq over \mathbb{N}_0
- ▶ $R = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$

German: Halbordnung oder partielle Ordnung

Least and Greatest Element

Definition (Least and greatest element)

Let \preceq be a partial order over set S .

An element $x \in S$ is the **least element** of S if **for all** $y \in S$ it holds that $x \preceq y$.

It is the **greatest element** of S if **for all** $y \in S$, $y \preceq x$.

- ▶ Is there a least/greatest element? Which one?
 - ▶ $S = \{1, 2, 3\}$ and $\preceq = \{(x, y) \mid x, y \in S \text{ and } x \leq y\}$
 - ▶ relation \leq over \mathbb{N}_0
 - ▶ relation \leq over \mathbb{Z}
- ▶ Why can we say **the** least element instead of **a** least element?

German: kleinstes/grösstes Element

Uniqueness of Least Element

Theorem

Let \preceq be a partial order over set S .

If S contains a least element, it contains exactly one least element.

Proof.

By contradiction: Assume x, y are least elements of S with $x \neq y$.

Since x is a least element, $x \preceq y$ is true.

Since y is a least element, $y \preceq x$ is true.

As a partial order is antisymmetric, this implies that $x = y$. ζ \square

Analogously: If there is a greatest element then is unique.

Minimal and Maximal Elements

Definition (Minimal/Maximal element of a set)

Let \preceq be a partial order over set S .

An element $x \in S$ is a **minimal element** of S
if **there is no $y \in S$ with $y \preceq x$ and $x \neq y$.**

An element $x \in S$ is a **maximal element** of S
if **there is no $y \in S$ with $x \preceq y$ and $x \neq y$.**

A set can have several minimal elements and no least element.

Example?

German: minimales/maximales Element

Total Orders

- ▶ Relations \leq over \mathbb{N}_0 and \subseteq for sets are partial orders.
- ▶ Can we compare every object against every object?
 - ▶ For all $x, y \in \mathbb{N}_0$ it holds that $x \leq y$ or that $y \leq x$ (or both).
 - ▶ $\{1, 2\} \not\subseteq \{2, 3\}$ and $\{2, 3\} \not\subseteq \{1, 2\}$
- ▶ Relation \leq is a **total** order, relation \subseteq is not.

Total Order – Definition

Definition (Total relation)

A binary relation R over set S is **total** if for all $x, y \in S$ at least one of xRy or yRx is true.

Definition (Total order)

A binary relation is a **total order** if it is **total** and a **partial order**.

German: totale Relation, (schwache) Totalordnung oder totale Ordnung

Strict Orders

- ▶ A **partial** order is reflexive, antisymmetric and transitive.
- ▶ We now consider **strict orders**.
- ▶ Example strict order relations are $<$ over \mathbb{N}_0 or \subset for sets.
- ▶ Are these relations
 - ▶ reflexive?
 - ▶ irreflexive?
 - ▶ symmetric?
 - ▶ asymmetric?
 - ▶ antisymmetric?
 - ▶ transitive?

Strict Orders – Definition

Definition (Strict (partial) order)

A binary relation \prec over set S is a **strict (partial) order** if \prec is **irreflexive, asymmetric and transitive**.

Which of these relations are strict orders?

- ▶ subset relation \subseteq for sets
- ▶ strict superset relation \supset for sets

Can a relation be both, a partial order and a strict (partial) order?

We can omit irreflexivity or asymmetry from the definition (but not both). Why?

German: strenge (Halb-)ordnung

Strict Total Orders

- ▶ As partial orders, a strict order does not automatically allow us to rank arbitrary two objects against each other.

- ▶ **Example 1** (personal preferences):

- ▶ “Pasta tastes better than potato.”

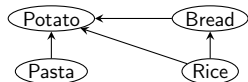
- ▶ “Rice tastes better than bread.”

- ▶ “Bread tastes better than potato.”

- ▶ “Rice tastes better than potato.”

- ▶ This definition of “tastes better than” is a strict order.

- ▶ No ranking of pasta against rice or of pasta against bread.



- ▶ **Example 2:** \subset relation for sets

- ▶ It **doesn't work** to simply require that the strict order is total.
Why?

Strict Total Orders – Definition

Definition (Trichotomy)

A binary relation R over set S is **trichotomous** if for all $x, y \in S$ exactly one of xRy , yRx or $x = y$ is true.

Definition (Strict total order)

A binary relation \prec over S is a **strict total order** if \prec is **trichotomous** and a **strict order**.

A strict total order completely ranks the elements of set S .

Example: $<$ relation over \mathbb{N}_0 gives the standard ordering $0, 1, 2, 3, \dots$ of natural numbers.

Attention: a non-empty strict total order is never a total order.

German: trichotom, strenge Totalordnung

Special Elements

Special elements are defined almost as for partial orders:

Definition (Least/greatest/minimal/maximal element of a set)

Let \prec be a **strict** order over set S .

An element $x \in S$ is the **least element** of S
if for all $y \in S$ where $y \neq x$ it holds that $x \prec y$.

It is the **greatest element** of S if for all $y \in S$ where $y \neq x$, $y \prec x$.

Element $x \in S$ is a **minimal element** of S
if there is no $y \in S$ with $y \prec x$.

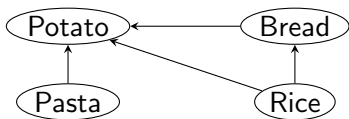
It is a **maximal element** of S
if there is no $y \in S$ with $x \prec y$.

Special Elements – Example

Consider again the previous example:

$$S = \{\text{Pasta, Potato, Bread, Rice}\}$$

$$\prec = \{(\text{Pasta, Potato}), (\text{Bread, Potato}), \\ (\text{Rice, Potato}), (\text{Rice, Bread})\}$$



Is there a least and a greatest element?

Which elements are maximal or minimal?

Summary

- ▶ An **equivalence relation** is **reflexive, symmetric and transitive**.
- ▶ A **partial order** $x \preceq y$ is **reflexive, antisymmetric and transitive**.
 - ▶ If x is the **greatest element** of a set S , it is greater than every element: for all $y \in S$ it holds that $y \preceq x$.
 - ▶ If x is a **maximal element** of set S then it is not smaller than any other element y : there is no $y \in S$ with $x \preceq y$ and $x \neq y$.
 - ▶ A **total order** is a partial order without incomparable objects.
- ▶ A **strict order** is **irreflexive, asymmetric and transitive**.
 - ▶ Strict **total orders** and **special elements** are analogously defined as for partial orders but with a special treatment of equal elements.