Discrete Mathematics in Computer Science

B2. Relations

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Relations

Relations: Informally

- Intuitively, a mathematical relation connects elements from several (possibly different) sets by specifying related groupings.
- We already know some relations, e. g.
  - $\subseteq$ relation for sets
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- These are examples of binary relations, considering pairs of objects.
- There are also relations of higher arity, e. g.
  - "$x + y = z$" for integers x, y, z.
  - "The name, address and office number belong to the same person."

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  - "$x + y = z$" for integers x, y, z.
  - "The name, address and office number belong to the same person."
- Relations are for example important for relational databases, semantic networks or knowledge representation and reasoning.

Relations

Definition (Relation)

Let $S_1, \dots, S_n$ be sets.

A relation over $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.

The arity of R is n.

A relation of arity n is a set of n-tuples.

German: Relation, Stelligkeit

Relations: Examples

- $\subseteq \; = \{(S, S') \mid S$ and S' are sets and
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- $R = \{(x, y, z) \mid x, y, z \in \mathbb{Z}$ and $x + y = z\}$

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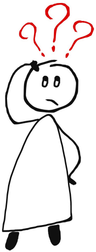
- $R = \{(x, y, z) \mid x, y, z \in \mathbb{Z} \text{ and } x + y = z\}$
- $R' = \{$(Gabi Röger, Spiegelgasse 1, 04.005),
  (Malte Helmert, Spiegelgasse 1, 06.004),
  (Florian Pommerening, Spiegelgasse 1, 04.005)$\}$

Questions



Questions?

Properties of Binary Relations

Binary Relation

A binary relation is a relation of arity 2:

Definition (binary relation)

A binary relation is a relation over two sets A and B.

German: zweistellige Relation, homogene Relation

Binary Relation

A binary relation is a relation of arity 2:

Definition (binary relation)

A binary relation is a relation over two sets A and B.

- Instead of $(x, y) \in R$, we also write xRy, e. g.
  $x \leq y$ instead of $(x, y) \in \leq$
- If the sets are equal, we say "R is a binary relation over A" instead of "R is a binary relation over A and A".
- Such a relation over a set is also called a homogeneous relation or an endorelation.

German: zweistellige Relation, homogene Relation

Reflexivity

A reflexive relation relates every object to itself.

Definition (reflexive)

A binary relation R over set A is reflexive
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How do these properties relate to irreflexivity?

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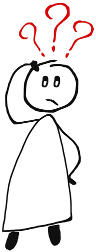

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- A binary relation is a relation over two sets.
- A binary relation over set S is a relation $R \subseteq S \times S$ and also called a homogeneous relation.
- A binary relation R over A is
  - reflexive if $(a, a) \in R$ for all $a \in A$,
  - irreflexive if $(a, a) \notin R$ for all $a \in A$,
  - symmetric if for all $a, b \in A$ it holds that $(a, b) \in R$ iff $(b, a) \in R$,
  - asymmetric if for all $a, b \in A$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$,
  - antisymmetric if for all $a, b \in A$ with $a \neq b$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$,
  - transitive if for all $a, b, c \in A$ it holds that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Special Classes of Relations

- Some important classes of relations are defined in terms of these properties.
  - Equivalence relation: reflexive, symmetric, transitive
  - Partial order: reflexive, antisymmetric, transitive
  - Strict order: irreflexive, asymmetric, transitive
  - ...
- We will consider these and other classes in detail.