# Discrete Mathematics in Computer Science

## B2. Relations

Malte Helmert, Gabriele Röger

University of Basel

October 7, 2024



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# B2.1 Relations





## Relations: Informally

- Intuitively, a mathematical relation connects elements from several (possibly different) sets by specifying related groupings.
- We already know some relations, e. g.
  - $\subseteq$ relation for sets
  - $\leq$ relation for natural numbers
- These are examples of binary relations, considering pairs of objects.
- There are also relations of higher arity, e. g.
  - "$x + y = z$" for integers $x, y, z$.
  - "The name, address and office number belong to the same person."
- Relations are for example important for relational databases, semantic networks or knowledge representation and reasoning.

## Relations

**Definition (Relation)**

Let $S_1, \dots, S_n$ be sets.

A relation over $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.

The arity of $R$ is $n$.

A relation of arity $n$ is a set of $n$-tuples.

German: Relation, Stelligkeit





## Relations: Examples

- $\subseteq \; = \{(S, S') \mid S$ and $S'$ are sets and for every $x \in S$ it holds that $x \in S'\}$
- $\leq \; = \{(x, y) \mid x, y \in \mathbb{N}_0$ and $x < y$ or $x = y\}$
- $R = \{(x, y, z) \mid x, y, z \in \mathbb{Z}$ and $x + y = z\}$
- $R' = \{$(Gabi Röger, Spiegelgasse 1, 04.005), (Malte Helmert, Spiegelgasse 1, 06.004), (Florian Pommerening, Spiegelgasse 1, 04.005)$\}$





# B2.2 Properties of Binary Relations





## Binary Relation

A binary relation is a relation of arity 2:

**Definition (binary relation)**

A binary relation is a relation over two sets $A$ and $B$.

- Instead of $(x, y) \in R$, we also write $xRy$, e. g. $x \leq y$ instead of $(x, y) \in \; \leq$
- If the sets are equal, we say "$R$ is a binary relation over $A$" instead of "$R$ is a binary relation over $A$ and $A$".
- Such a relation over a set is also called a homogeneous relation or an endorelation.

German: zweistellige Relation, homogene Relation

## Reflexivity

A reflexive relation relates every object to itself.

**Definition (reflexive)**
A binary relation $R$ over set $A$ is reflexive
if for all $a \in A$ it holds that $(a, a) \in R$.

Which of these relations are reflexive?

- $R = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$ over $\{a, b, c\}$
- $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- equality relation $=$ on natural numbers
- less-than relation $\leq$ on natural numbers
- strictly-less-than relation $<$ on natural numbers

German: reflexiv





## Irreflexivity

A irreflexive relation never relates an object to itself.

**Definition (irreflexive)**
A binary relation $R$ over set $A$ is irreflexive
if for all $a \in A$ it holds that $(a, a) \notin R$.

Which of these relations are irreflexive?

- $R = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$ over $\{a, b, c\}$
- $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- equality relation $=$ on natural numbers
- less-than relation $\leq$ on natural numbers
- strictly-less-than relation $<$ on natural numbers

German: irreflexiv





## Symmetry

**Definition (symmetric)**
A binary relation $R$ over set $A$ is symmetric
if for all $a, b \in A$ it holds that $(a, b) \in R$ iff $(b, a) \in R$.

Which of these relations are symmetric?

- $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- equality relation $=$ on natural numbers
- less-than relation $\leq$ on natural numbers
- strictly-less-than relation $<$ on natural numbers

German: symmetrisch





## Asymmetry and Antisymmetry

How do these properties relate to irreflexivity?

**Definition (asymmetric and antisymmetric)**
Let $R$ be a binary relation over set $A$.

Relation $R$ is asymmetric if
for all $a, b \in A$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$.

Relation $R$ is antisymmetric if for all $a, b \in A$ with $a \neq b$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$.

Which of these relations are asymmetric/antisymmetric?

- $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- equality relation $=$ on natural numbers
- less-than relation $\leq$ on natural numbers
- strictly-less-than relation $<$ on natural numbers

German: asymmetrisch, antisymmetrisch

## Transitivity

**Definition**

A binary relation $R$ over set $A$ is transitive if it holds for all $a, b, c \in A$ that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Which of these relations are transitive?

- $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- equality relation $=$ on natural numbers
- less-than relation $\leq$ on natural numbers
- strictly-less-than relation $<$ on natural numbers

German: transitiv





## Summary

- A relation over sets $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.
- A binary relation is a relation over two sets.
- A binary relation over set $S$ is a relation $R \subseteq S \times S$ and also called a homogeneous relation.
- A binary relation $R$ over $A$ is
  - reflexive if $(a, a) \in R$ for all $a \in A$,
  - irreflexive if $(a, a) \notin R$ for all $a \in A$,
  - symmetric if for all $a, b \in A$ it holds that $(a, b) \in R$ iff $(b, a) \in R$,
  - asymmetric if for all $a, b \in A$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$,
  - antisymmetric if for all $a, b \in A$ with $a \neq b$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$,
  - transitive if for all $a, b, c \in A$ it holds that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.





## Special Classes of Relations

- Some important classes of relations are defined in terms of these properties.
  - Equivalence relation: reflexive, symmetric, transitive
  - Partial order: reflexive, antisymmetric, transitive
  - Strict order: irreflexive, asymmetric, transitive
  - ...
- We will consider these and other classes in detail.