Discrete Mathematics in Computer Science

B1. Tuples & Cartesian Product

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B1.1 Tuples and the Cartesian Product

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Motivation

- A set is an unordered collection of distinct objects.
- We often need a more structured way of representation.
  - A person is associated with a name, address, phone number.
  - A set of persons makes sense in many contexts.
  - Representing the associated data as a set rather not.
- We could for example want to
  - directly access the name of a person, or
  - have a separate billing and delivery address for some order, but in general, these can be the same.
- Tuples are mathematical building blocks that support this.

Sets vs. Tuples

- A set is an unordered collection of distinct objects.
- A tuple is an ordered sequence of objects.

Tuples

- k-tuple: ordered sequence of k objects ($k \in \mathbb{N}_0$)
- written $(o_1, \dots, o_k)$ or $\langle o_1, \dots, o_k \rangle$
- unlike sets, order matters ($\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$)
- objects may occur multiple times in a tuple
- objects contained in tuples are called components
- terminology:
  - $k = 2$: (ordered) pair
  - $k = 3$: triple
  - more rarely: quadruple, quintuple, sextuple, septuple, ...
- if k is clear from context (or does not matter), often just called tuple

German: k-Tupel, Komponente, (geordnetes) Paar, Tripel, Quadrupel

Equality of Tuples

Definition (Equality of Tuples)

Two n-tuples $t = \langle o_1, \dots, o_n \rangle$ and $t' = \langle o'_1, \dots, o'_n \rangle$ are equal ($t = t'$) if for $i \in \{1, \dots, n\}$ it holds that $o_i = o'_i$.

Cartesian Product

Definition (Cartesian Product and Cartesian Power)

Let $S_1, \dots, S_n$ be sets. The Cartesian product $S_1 \times \dots \times S_n$ is the following set of n-tuples:

$$S_1 \times \dots \times S_n = \{\langle x_1, \dots, x_n \rangle \mid x_1 \in S_1, x_2 \in S_2, \dots, x_n \in S_n\}.$$

The k-ary Cartesian power of a set S (with $k \in \mathbb{N}_1$) is the set $S^k = \{\langle o_1, \dots, o_k \rangle \mid o_i \in S \text{ for all } i \in \{1, \dots, k\}\} = \underbrace{S \times \dots \times S}_{k \text{ times}}$.

René Descartes: French mathematician and philosopher (1596–1650)

Example: $A = \{a, b\}$, $B = \{1, 2, 3\}$

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$
$$A^2 = \{(a, a), (a, b), (b, a), (b, b)\}$$

German: Kartesisches Produkt

(Non-)properties of the Cartesian Product

The Cartesian product is

- not commutative, in most cases $A \times B \neq B \times A$.
- not associative, in most cases $(A \times B) \times C \neq A \times (B \times C)$

Why? Exceptions?

Summary

- A k-tuple is an ordered sequence of k objects, called the components of the tuple.
- 2-tuples are also called pairs and 3-tuples triples.
- The Cartesian Product $S_1 \times \cdots \times S_n$ of set $S_1, \ldots, S_n$ is the set of all tuples $\langle o_1, \ldots, o_n \rangle$, where for all $i \in \{1, \ldots, n\}$ component o_i is an element of S_i.