Discrete Mathematics in Computer Science

A4. Proof Techniques I

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Proof Strategies

Common Forms of Statements

Many statements have one of these forms:

1. "All $x \in S$ with the property P also have the property Q."
2. "A is a subset of B."
3. "For all $x \in S$: x has property P iff x has property Q."
4. "$A = B$", where A and B are sets.

Common Forms of Statements

Many statements have one of these forms:

1. "All $x \in S$ with the property P also have the property Q."
2. "A is a subset of B."
3. "For all $x \in S$: x has property P iff x has property Q."
4. "$A = B$", where A and B are sets.

In the following, we will discuss some typical proof/disproof strategies for such statements.

Proof Strategies

1. "All $x \in S$ with the property P also have the property Q."
   "For all $x \in S$: if x has property P, then x has property Q."
   - To prove, assume you are given an arbitrary $x \in S$ that has the property P.
     Give a sequence of proof steps showing that x must have the property Q.
   - To disprove, find a counterexample, i. e., find an $x \in S$ that has property P but not Q and prove this.

Proof Strategies

2. "A is a subset of B."
   - To prove, assume you have an arbitrary element $x \in A$ and prove that $x \in B$.
   - To disprove, find an element in $x \in A \setminus B$ and prove that $x \in A \setminus B$.

Proof Strategies

3. "For all $x \in S$: x has property P iff x has property Q."
   ("iff": "if and only if")
   - To prove, separately prove "if P then Q" and "if Q then P".
   - To disprove, disprove "if P then Q" or disprove "if Q then P".

Proof Strategies

4. "$A = B$", where A and B are sets.
   - To prove, separately prove "$A \subseteq B$" and "$B \subseteq A$".
   - To disprove, disprove "$A \subseteq B$" or disprove "$B \subseteq A$".

Proof Techniques

most common proof techniques:

- direct proof
- indirect proof (proof by contradiction)
- contrapositive
- mathematical induction
- structural induction

Direct Proof

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Direct Proof

Direct derivation of the statement by deducing or rewriting.

German: Direkter Beweis

Direct Proof: Example

Theorem

For all sets A, B and C it holds that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof.

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Proof.

Let A, B and C be arbitrary sets.

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Theorem

For all sets A, B and C it holds that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof.

Let A, B and C be arbitrary sets.

We will show separately that

- $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and that
- $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

...

Direct Proof: Example cont.

Proof (continued).

We first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:

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Proof (continued).

We first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:

If $A \cap (B \cup C)$ is empty, the statement is trivially true. Otherwise consider an arbitrary $x \in A \cap (B \cup C)$.

Direct Proof: Example cont.

Proof (continued).

We first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:

If $A \cap (B \cup C)$ is empty, the statement is trivially true. Otherwise consider an arbitrary $x \in A \cap (B \cup C)$. By the definition of the intersection it holds that $x \in A$ and that $x \in (B \cup C)$.

Direct Proof: Example cont.

Proof (continued).

We first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:

If $A \cap (B \cup C)$ is empty, the statement is trivially true. Otherwise consider an arbitrary $x \in A \cap (B \cup C)$. By the definition of the intersection it holds that $x \in A$ and that $x \in (B \cup C)$.
We make a case distinction between $x \in B$ and $x \notin B$:

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Proof (continued).

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If $A \cap (B \cup C)$ is empty, the statement is trivially true. Otherwise consider an arbitrary $x \in A \cap (B \cup C)$. By the definition of the intersection it holds that $x \in A$ and that $x \in (B \cup C)$.
We make a case distinction between $x \in B$ and $x \notin B$:

Case 1 ($x \in B$): As $x \in A$ is true, it holds in this case that $x \in (A \cap B)$.

Direct Proof: Example cont.

Proof (continued).

We first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:

If $A \cap (B \cup C)$ is empty, the statement is trivially true. Otherwise consider an arbitrary $x \in A \cap (B \cup C)$. By the definition of the intersection it holds that $x \in A$ and that $x \in (B \cup C)$.
We make a case distinction between $x \in B$ and $x \notin B$:

Case 1 ($x \in B$): As $x \in A$ is true, it holds in this case that $x \in (A \cap B)$.

Case 2 ($x \notin B$): From $x \in (B \cup C)$ it follows for this case that $x \in C$. With $x \in A$ we conclude that $x \in (A \cap C)$.

Direct Proof: Example cont.

Proof (continued).

We first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:

If $A \cap (B \cup C)$ is empty, the statement is trivially true. Otherwise consider an arbitrary $x \in A \cap (B \cup C)$. By the definition of the intersection it holds that $x \in A$ and that $x \in (B \cup C)$.
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Case 1 ($x \in B$): As $x \in A$ is true, it holds in this case that $x \in (A \cap B)$.

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In both cases it holds that $x \in A \cap B$ or $x \in A \cap C$, and we conclude that $x \in (A \cap B) \cup (A \cap C)$.

Direct Proof: Example cont.

Proof (continued).

We first show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:

If $A \cap (B \cup C)$ is empty, the statement is trivially true. Otherwise consider an arbitrary $x \in A \cap (B \cup C)$. By the definition of the intersection it holds that $x \in A$ and that $x \in (B \cup C)$.
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In both cases it holds that $x \in A \cap B$ or $x \in A \cap C$, and we conclude that $x \in (A \cap B) \cup (A \cap C)$.

As x was chosen arbitrarily from $A \cap (B \cup C)$, we have shown that every element of $A \cap (B \cup C)$ is an element of $(A \cap B) \cup (A \cap C)$, so it holds that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. ...

Direct Proof: Example cont.

Proof (continued).

We will now show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

... **[Homework assignment]** ...

Overall we have shown for arbitrary sets A, B and C that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, which concludes the proof of the theorem. □

Indirect Proof

Indirect Proof

Indirect Proof (Proof by Contradiction)

- Make an assumption that the statement is false.
- Use the assumption to derive a contradiction.
- This shows that the assumption must be false
  and hence the original statement must be true.

German: Indirekter Beweis, Beweis durch Widerspruch

Indirect Proof: Example

Theorem

Let A and B be sets. If $A \setminus B = \emptyset$ then $A \subseteq B$.

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Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \not\subseteq B$.

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Proof.

We prove the theorem by contradiction.

Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \not\subseteq B$.

Let A and B be such sets.

Indirect Proof: Example

Theorem

Let A and B be sets. If $A \setminus B = \emptyset$ then $A \subseteq B$.

Proof.

We prove the theorem by contradiction.

Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \nsubseteq B$.

Let A and B be such sets.

Since $A \nsubseteq B$ there is some $x \in A$ such that $x \notin B$.

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Let A and B be sets. If $A \setminus B = \emptyset$ then $A \subseteq B$.

Proof.

We prove the theorem by contradiction.

Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \nsubseteq B$.

Let A and B be such sets.

Since $A \nsubseteq B$ there is some $x \in A$ such that $x \notin B$.

For this x it holds that $x \in A \setminus B$.

Indirect Proof: Example

Theorem

Let A and B be sets. If $A \setminus B = \emptyset$ then $A \subseteq B$.

Proof.

We prove the theorem by contradiction.

Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \not\subseteq B$.

Let A and B be such sets.

Since $A \not\subseteq B$ there is some $x \in A$ such that $x \notin B$.

For this x it holds that $x \in A \setminus B$.

This is a contradiction to $A \setminus B = \emptyset$.

We conclude that the assumption was false and thus the theorem is true. □

Proof by Contrapositive

Contrapositive

(Proof by) Contrapositive

Prove "If A, then B" by proving "If not B, then not A."

Contrapositive

(Proof by) Contrapositive

Prove "If A, then B" by proving "If not B, then not A."

Examples:

- Prove "For all $n \in \mathbb{N}_0$: if n^2 is odd, then n is odd" by proving "For all $n \in \mathbb{N}_0$, if n is even, then n^2 is even."
- Prove "For all $n \in \mathbb{N}_0$: if n is not a square number, then $\sqrt{n}$ is irrational" by proving "For all $n \in \mathbb{N}_0$: if $\sqrt{n}$ is rational, then n is a square number."

German: Kontraposition

Contrapositive: Example

Theorem

For any sets A and B: If $A \subseteq B$ then $A \setminus B = \emptyset$.

Contrapositive: Example

Theorem

For any sets A and B: If $A \subseteq B$ then $A \setminus B = \emptyset$.

Proof.

We prove the theorem by contrapositive, showing for any sets A and B that if $A \setminus B \neq \emptyset$ then $A \not\subseteq B$.

Contrapositive: Example

Theorem

For any sets A and B: If $A \subseteq B$ then $A \setminus B = \emptyset$.

Proof.

We prove the theorem by contrapositive, showing for any sets A and B that if $A \setminus B \neq \emptyset$ then $A \not\subseteq B$.

Let A and B be arbitrary sets with $A \setminus B \neq \emptyset$.

Contrapositive: Example

Theorem

For any sets A and B: If $A \subseteq B$ then $A \setminus B = \emptyset$.

Proof.

We prove the theorem by contrapositive, showing for any sets A and B that if $A \setminus B \neq \emptyset$ then $A \not\subseteq B$.

Let A and B be arbitrary sets with $A \setminus B \neq \emptyset$.

As the set difference is not empty, there is at least one x with $x \in A \setminus B$.

Contrapositive: Example

Theorem

For any sets A and B: If $A \subseteq B$ then $A \setminus B = \emptyset$.

Proof.

We prove the theorem by contrapositive, showing for any sets A and B that if $A \setminus B \neq \emptyset$ then $A \not\subseteq B$.

Let A and B be arbitrary sets with $A \setminus B \neq \emptyset$.

As the set difference is not empty, there is at least one x with $x \in A \setminus B$. By the definition of the set difference ($\setminus$), it holds for such x that $x \in A$ and $x \notin B$.

Contrapositive: Example

Theorem

For any sets A and B: If $A \subseteq B$ then $A \setminus B = \emptyset$.

Proof.

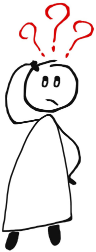
We prove the theorem by contrapositive, showing for any sets A and B that if $A \setminus B \neq \emptyset$ then $A \not\subseteq B$.

Let A and B be arbitrary sets with $A \setminus B \neq \emptyset$.

As the set difference is not empty, there is at least one x with $x \in A \setminus B$. By the definition of the set difference ($\setminus$), it holds for such x that $x \in A$ and $x \notin B$.

Hence, not all elements of A are elements of B, so it does not hold that $A \subseteq B$. □

Questions



Questions?

Summary

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- There are standard strategies for proving some common forms of statements, e.g. some property of all elements of a set.
- Direct proof: derive statement by deducing or rewriting.
- Indirect proof: derive contradiction from the assumption that the statement is false.
- Proof by contrapositive: Prove "If A, then B" by proving "If not B, then not A.".