

Discrete Mathematics in Computer Science

A3. Proofs: Introduction

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September 23, 2024

What is a Proof?

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- a sequence of logical steps
- starting with one set of statements
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What is a **statement**?

Mathematical Statements

Mathematical Statement

A **mathematical statement** consists of a set of **preconditions** and a set of **conclusions**.

The statement is **true** if the conclusions are true whenever the preconditions are true.

The set of preconditions is sometimes empty.

German: Mathematische Aussage

Examples of Mathematical Statements

Examples (some true, some false):

- “Let $p \in \mathbb{N}_0$ be a prime number. Then p is odd.”
- “There exists an even prime number.”
- “Let $p \in \mathbb{N}_0$ be a prime number with $p \geq 3$. Then p is odd.”
- “All prime numbers $p \geq 3$ are odd.”
- “If 4 is a prime number then $2 \cdot 3 = 4$.”

What are the preconditions, what are the conclusions?

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What are the preconditions, what are the conclusions?

Which ones are true, which ones are false?

On what Statements can we Build the Proof?

A mathematical proof is

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that some statement must be true.

We can use:

- **axioms**: statements that are assumed to always be true
in the current context
- **theorems** and **lemmas**: statements that were already proven
 - lemma: an intermediate tool
 - theorem: itself a relevant result
- **premises**: assumptions we make
to see what consequences they have

German: Axiom, Theorem/Satz, Lemma, Prämisse/Annahme

What is a Logical Step?

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Each step directly follows

- from the axioms,
- premises,
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For a formal definition, we would need formal logics.

The Role of Definitions

Definition

A **set** is an unordered collection of distinct objects.

The objects in a set are called the **elements** of the set. A set is said to **contain** its elements.

We write $x \in S$ to indicate that x is an element of set S , and $x \notin S$ to indicate that S does not contain x .

The set that does not contain any objects is the **empty set** \emptyset .

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- A definition introduces an abbreviation.
- Whenever we say “set”, we could instead say “an unordered collection of distinct objects” and vice versa.
- Definitions can also introduce notation.

German: Definition

Disproofs

- A **disproof** (**refutation**) shows that a given mathematical statement is **false** by giving an example where the preconditions are true, but the conclusion is false.
- This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.

German: Widerlegung

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Example (False statement)

"If $p \in \mathbb{N}_0$ is a prime number then p is odd."

Refutation.

Consider natural number 2 as a counter example. It is prime because it has exactly 2 divisors, 1 and itself. It is not odd, because it is divisible by 2. □

German: Widerlegung

A Word on Style

A proof should help the reader to see why the result must be true.

- A proof should be easy to follow.
- Omit unnecessary information.
- Move self-contained parts into separate lemmas.
- In complicated proofs, reveal the overall structure in advance.
- Have a clear line of argument.

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Recommended reading (ADAM additional resources):

- “Some Remarks on Writing Mathematical Proofs” (John M. Lee)
- “§1. Minicourse on technical writing” of “Mathematical Writing” (Donald E. Knuth, Tracy Larrabee, and Paul M. Roberts)

Questions



Questions?

Summary

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A proof should convince the reader by **logical steps** of the truth of some mathematical statement.