

# Discrete Mathematics in Computer Science

A2. Sets: Foundations

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A2.1 Sets

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## A2.1 Sets

## Important Building Blocks of Discrete Mathematics

- ▶ sets
- ▶ relations
- ▶ functions

These topics will mainly be the content of part B of the course.  
We cover some foundations on sets already now because we will use them for illustrating proof techniques.

## Sets

### Definition

A **set** is an **unordered collection** of **distinct** objects.

- ▶ **unordered**: no notion of a “first” or “second” object,  
e. g.  $\{Alice, Bob, Charly\} = \{Charly, Bob, Alice\}$
- ▶ **distinct**: each object contained **at most once**,  
e. g.  $\{Alice, Bob, Charly\} = \{Alice, Charly, Bob, Alice\}$

German: Menge

## Notation

- ▶ Specification of sets
  - ▶ **explicit**, listing all elements, e. g.  $A = \{1, 2, 3\}$
  - ▶ **implicit** with **set-builder notation**, specifying a **property** characterizing all elements,  
e. g.  $A = \{x \mid x \in \mathbb{N}_0 \text{ and } 1 \leq x \leq 3\}$ ,  
 $B = \{n^2 \mid n \in \mathbb{N}_0\}$
  - ▶ **implicit**, as a **sequence with dots**,  
e. g.  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
  - ▶ **implicit** with an **inductive definition**
- ▶  $e \in M$ :  $e$  is in set  $M$  (an **element** of the set)
- ▶  $e \notin M$ :  $e$  is not in set  $M$
- ▶ **empty set**  $\emptyset = \{\}$

**Question:** Is it true that  $1 \in \{\{1, 2\}, 3\}$ ?

German: Element, leere Menge

## Special Sets

- ▶ **Natural numbers**  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$
- ▶ **Integers**  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- ▶ **Positive integers**  $\mathbb{Z}_+ = \mathbb{N}_1 = \{1, 2, \dots\}$
- ▶ **Rational numbers**  $\mathbb{Q} = \{n/d \mid n \in \mathbb{Z}, d \in \mathbb{N}_1\}$
- ▶ **Real numbers**  $\mathbb{R} = (-\infty, \infty)$   
Why do we use interval notation?  
Why didn't we introduce it before?

German: Natürliche ( $\mathbb{N}_0$ ), ganze ( $\mathbb{Z}$ ), rationale ( $\mathbb{Q}$ ), reelle ( $\mathbb{R}$ ) Zahlen

## A2.2 Russell's Paradox

## Excursus: Barber Paradox

### Barber Paradox

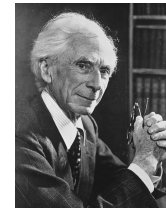
In a town there is only one barber, who is male.  
The barber shaves all men in the town,  
and only those, who do not shave themselves.

Who shaves the barber?



We can exploit the self-reference to derive a contradiction.

## Russell's Paradox



Bertrand Russell

### Question

Is the collection of all sets that do not contain themselves as a member a set?

Is  $S = \{M \mid M \text{ is a set and } M \notin M\}$  a set?

Assume that  $S$  is a set.

If  $S \notin S$  then  $S \in S \rightsquigarrow$  Contradiction

If  $S \in S$  then  $S \notin S \rightsquigarrow$  Contradiction

Hence, there is no such set  $S$ .

→ Not every property used in set-builder notation defines a set.

## A2.3 Relations on Sets

## Equality

### Definition (Axiom of Extensionality)

Two sets  $A$  and  $B$  are **equal** (written  $A = B$ )  
if every element of  $A$  is an element of  $B$  and vice versa.

Two sets are equal if they contain the same elements.

We write  $A \neq B$  to indicate that  $A$  and  $B$  are **not** equal.

## Subsets and Supersets

- ▶  $A \subseteq B$ :  $A$  is a **subset** of  $B$ ,  
i. e., every element of  $A$  is an element of  $B$
- ▶  $A \subset B$ :  $A$  is a **strict subset** of  $B$ ,  
i. e.,  $A \subseteq B$  and  $A \neq B$ .
- ▶  $A \supseteq B$ :  $A$  is a **superset** of  $B$  if  $B \subseteq A$ .
- ▶  $A \supset B$ :  $A$  is a **strict superset** of  $B$  if  $B \subset A$ .

We write  $A \not\subseteq B$  to indicate that  $A$  is **not** a subset of  $B$ .

Analogously:  $\not\subset$ ,  $\not\supseteq$ ,  $\not\supset$

German: Teilmenge, echte Teilmenge, Obermenge, echte Obermenge

## Power Set

### Definition (Power Set)

The **power set**  $\mathcal{P}(S)$  of a set  $S$  is the set of all subsets of  $S$ .  
That is,

$$\mathcal{P}(S) = \{M \mid M \subseteq S\}.$$

Example:  $\mathcal{P}(\{a, b\}) =$

German: Potenzmenge

## A2.4 Set Operations

## Set Operations

Set operations allow us to express sets in terms of other sets

- ▶ **intersection**  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



If  $A \cap B = \emptyset$  then  $A$  and  $B$  are **disjoint**.

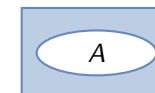
- ▶ **union**  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



- ▶ **set difference**  $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



- ▶ **complement**  $\bar{A} = B \setminus A$ , where  $A \subseteq B$  and  $B$  is the set of all considered objects (in a given context)



German: Schnitt, disjunkt, Vereinigung, Differenz, Komplement

## Properties of Set Operations: Commutativity

### Theorem (Commutativity of $\cup$ and $\cap$ )

For all sets  $A$  and  $B$  it holds that

- ▶  $A \cup B = B \cup A$  and
- ▶  $A \cap B = B \cap A$ .

**Question:** Is the set difference also commutative,  
i. e. is  $A \setminus B = B \setminus A$  for all sets  $A$  and  $B$ ?

German: Kommutativität

## Properties of Set Operations: Associativity

### Theorem (Associativity of $\cup$ and $\cap$ )

For all sets  $A, B$  and  $C$  it holds that

- ▶  $(A \cup B) \cup C = A \cup (B \cup C)$  and
- ▶  $(A \cap B) \cap C = A \cap (B \cap C)$ .

German: Assoziativität

## Properties of Set Operations: Distributivity

### Theorem (Union distributes over intersection and vice versa)

For all sets  $A, B$  and  $C$  it holds that

- ▶  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  and
- ▶  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

German: Distributivität

## Properties of Set Operations: De Morgan's Law



Augustus De Morgan  
British mathematician (1806-1871)

### Theorem (De Morgan's Law)

For all sets  $A$  and  $B$  it holds that

- ▶  $\overline{A \cup B} = \overline{A} \cap \overline{B}$  and
- ▶  $\overline{A \cap B} = \overline{A} \cup \overline{B}$ .

## A2.5 Cardinality of Finite Sets

## Cardinality of Sets

The **cardinality**  $|S|$  measures the size of set  $S$ .

A set is **finite** if it has a finite number of elements.

### Definition (Cardinality)

The **cardinality** of a finite set is the **number of elements** it contains.

- ▶  $|\emptyset| =$
- ▶  $|\{x \mid x \in \mathbb{N}_0 \text{ and } 2 \leq x < 5\}| =$
- ▶  $|\{3, 0, \{1, 3\}\}| =$
- ▶  $|\mathcal{P}(\{1, 2\})| =$

German: Kardinalität oder Mächtigkeit

## Cardinality of the Union of Sets

### Theorem

For finite sets  $A$  and  $B$  it holds that  $|A \cup B| = |A| + |B| - |A \cap B|$ .

### Corollary

If finite sets  $A$  and  $B$  are **disjoint** then  $|A \cup B| = |A| + |B|$ .

## Cardinality of the Power Set

### Theorem

Let  $S$  be a finite set. Then  $|\mathcal{P}(S)| = 2^{|S|}$ .

### Proof sketch.

We can construct a subset  $S'$  by iterating over all elements  $e$  of  $S$  and deciding whether  $e$  becomes a member of  $S'$  or not.

We make  $|S|$  independent decisions, each between two options. Hence, there are  $2^{|S|}$  possible outcomes.

Every subset of  $S$  can be constructed this way and different choices lead to different sets. Thus,  $|\mathcal{P}(S)| = 2^{|S|}$ .  $\square$

## Summary

- ▶ Sets are **unordered collections** of **distinct** objects.
- ▶ Important **set relations**: **equality** ( $=$ ), **subset** ( $\subseteq$ ), **superset** ( $\supseteq$ ) and strict variants ( $\subset$  and  $\supset$ )
- ▶ The **power set** of a set  $S$  is the set of all subsets of  $S$ .
- ▶ Important **set operations** are **intersection**, **union**, **set difference** and **complement**.
  - ▶ Union and intersection are **commutative and associative**.
  - ▶ Union distributes over intersection and vice versa.
  - ▶ **De Morgan's law** for complement of union or intersection.
- ▶ The number of elements in a finite set is called its **cardinality**.