Solution of Exercise in A5

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1 Basic definitions (from the slides)

The set $\mathcal{B}$ of binary trees is inductively defined as follows:

- $\square$ is a binary tree (a *leaf*)
- If L and R are binary trees, then $\langle L, \bigcirc, R\rangle$ is a binary tree (with *inner node* $\bigcirc$).

The number of *leaves* of a binary tree B, written $\mathit{leaves}(B)$, is defined as follows:

$$\begin{aligned} \mathit{leaves}(\square) &= 1 \\ \mathit{leaves}(\langle L, \bigcirc, R\rangle) &= \mathit{leaves}(L) + \mathit{leaves}(R) \end{aligned}$$

The *height* of a binary tree B, written $\mathit{height}(B)$, is defined as follows:

$$\begin{aligned} \mathit{height}(\square) &= 0 \\ \mathit{height}(\langle L, \bigcirc, R\rangle) &= \max\{\mathit{height}(L), \mathit{height}(R)\} + 1 \end{aligned}$$

2 Example: Structural Induction

We will establish the following theorem:

Theorem 1. *For all binary trees B: $\mathit{leaves}(B) \leq 2^{\mathit{height}(B)}$.*

Proof. We prove the theorem by structural induction.

Induction basis: If $B = \square$, we have $\mathit{leaves}(\square) = 1 = 2^0 = 2^{\mathit{height}(\square)}$. ✓

Induction hypothesis: Suppose that for binary trees L and R it holds that $\mathit{leaves}(L) \leq 2^{\mathit{height}(L)}$ and $\mathit{leaves}(R) \leq 2^{\mathit{height}(R)}$.

Inductive step: Consider $B = \langle L, \bigcirc, R\rangle$.

By the definition of *leaves*, it holds that $\mathit{leaves}(B) = \mathit{leaves}(L) + \mathit{leaves}(R)$. Using the induction hypothesis, we get

$$\begin{aligned}
\mathit{leaves}(B) &= \mathit{leaves}(L) + \mathit{leaves}(R) \\
&\overset{\text{IH}}{\leq} 2^{\mathit{height}(L)} + 2^{\mathit{height}(R)} \\
&\overset{(*)}{\leq} 2^{\max\{\mathit{height}(L), \mathit{height}(R)\}} + 2^{\max\{\mathit{height}(L), \mathit{height}(R)\}} \\
&= 2 \cdot 2^{\max\{\mathit{height}(L), \mathit{height}(R)\}} \\
&\overset{(**)}{=} 2 \cdot 2^{\mathit{height}(B) - 1} \\
&= 2^{\mathit{height}(B)}
\end{aligned}$$

For inequality $(*)$, we use that $\mathit{height}(L) \leq \max\{\mathit{height}(L), \mathit{height}(R)\}$ and $\mathit{height}(R) \leq \max\{\mathit{height}(L), \mathit{height}(R)\}$. For equality $(**)$, we use the definition of *height*. □