Planning and Optimization

G6. Linear & Integer Programming

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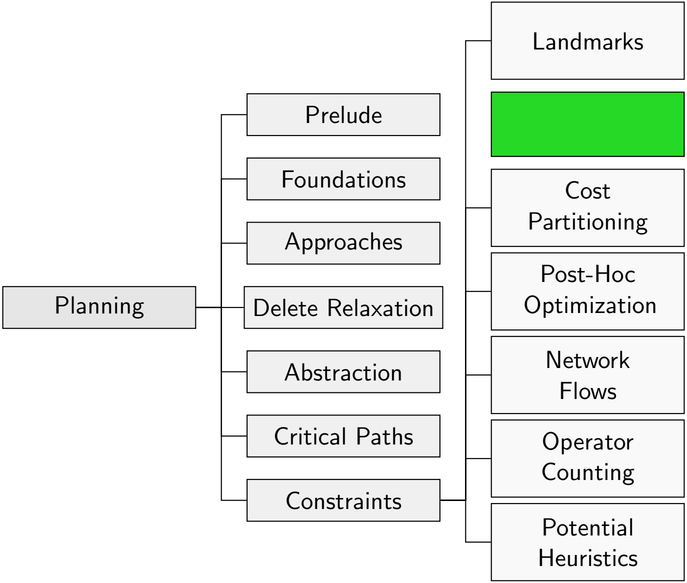
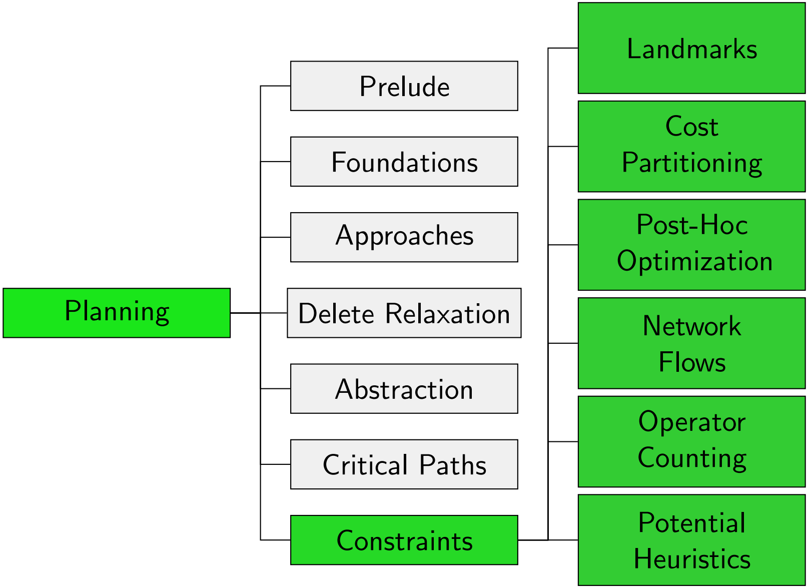
December 6, 2023

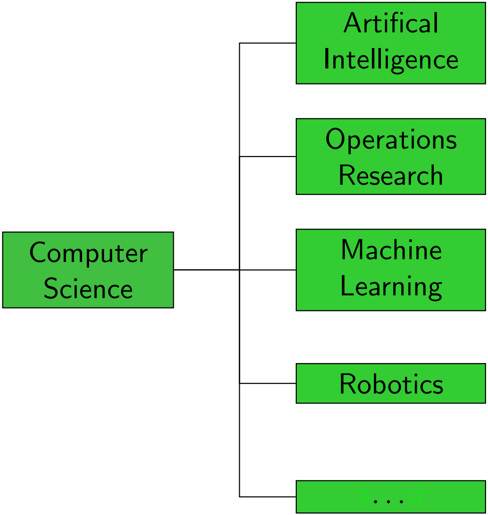
Content of this Course (Timeline)



Content of this Course (Relevance)



Not Content of this Course (Relevance)



Integer Programs

Motivation

- This goes on beyond Computer Science
- Active research on IPs and LPs in
  - Operation Research
  - Mathematics
- Many application areas, for instance:
  - Manufacturing
  - Agriculture
  - Mining
  - Logistics
  - Planning
- As an application, we treat LPs / IPs as a blackbox
- We just look at the fundamentals
- However, even on the application side there is much more (e.g., modelling tricks or solver parameters to speed up computation)

Motivation

Example (Optimization Problem)

Consider the following scenario:

- A factory produces two products A and B
- Selling one (unit of) B yields 5 times the profit of selling one A
- A client places the unusual order to "buy anything that can be produced on that day as long as two plus twice the units of A is not smaller than the number of B"
- More than 12 products in total cannot be produced per day
- There is only material for 6 units of A (there is enough material to produce any amount of B)

How many units of A and B does the client receive if the factory owner aims to maximize her profit?

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

$$X_A \geq 0, \quad X_B \geq 0$$

Example (Optimization Problem)

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

$$\text{maximize} \quad X_A + 5X_B \quad \text{subject to}$$

$$X_A \geq 0, \quad X_B \geq 0$$

Example (Optimization Problem)

- "one B yields 5 times the profit of one A"
- "the factory owner aims to maximize her profit"

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

$$\text{maximize} \quad X_A + 5X_B \quad \text{subject to}$$

$$2 + 2X_A \geq X_B$$

$$X_A \geq 0, \quad X_B \geq 0$$

Example (Optimization Problem)

- "two plus twice the units of A may not be smaller than the number of B"

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

$$\text{maximize} \quad X_A + 5X_B \quad \text{subject to}$$

$$2 + 2X_A \geq X_B$$
$$X_A + X_B \leq 12$$

$$X_A \geq 0, \quad X_B \geq 0$$

Example (Optimization Problem)

- "More than 12 products in total cannot be produced per day"

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

$$\text{maximize} \quad X_A + 5X_B \quad \text{subject to}$$

$$2 + 2X_A \geq X_B$$
$$X_A + X_B \leq 12$$
$$X_A \leq 6$$

$$X_A \geq 0, \quad X_B \geq 0$$

Example (Optimization Problem)

- “There is only material for 6 units of A”

Integer Program: Example

Let X_A and X_B be the (integer) number of produced A and B

Example (Optimization Problem as Integer Program)

maximize $X_A + 5X_B$ subject to

$$2 + 2X_A \geq X_B$$
$$X_A + X_B \leq 12$$
$$X_A \leq 6$$

$$X_A \geq 0, \quad X_B \geq 0$$

$\rightsquigarrow$ unique optimal solution:
produce 4 A ($X_A = 4$) and 8 B ($X_B = 8$) for a profit of 44

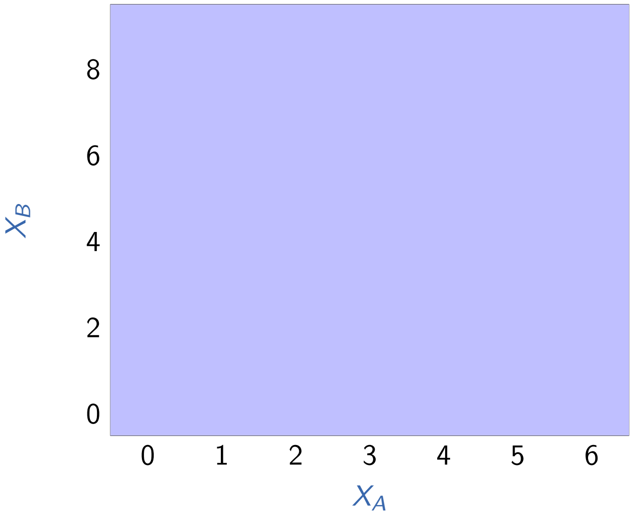
Same Program as Input for the CPLEX Solver

File ip.lp

```
Maximize
 obj: X_A + 5 X_B
Subject To
 c1: -2 X_A + X_B <= 2
 c2: X_A + X_B <= 12
Bounds
 0 <= X_A <= 6
 0 <= X_B
General
 X_A X_B
End
```

→ Demo

Integer Program Example: Visualization



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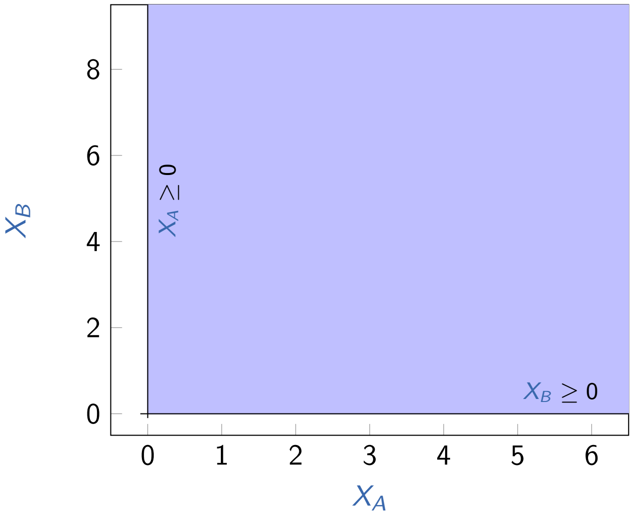

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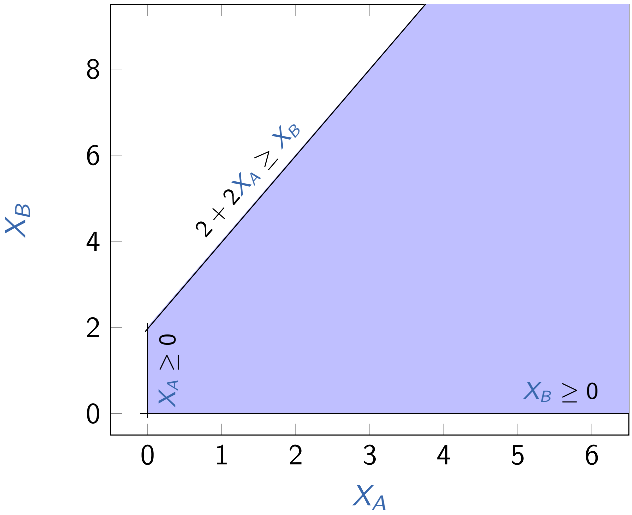

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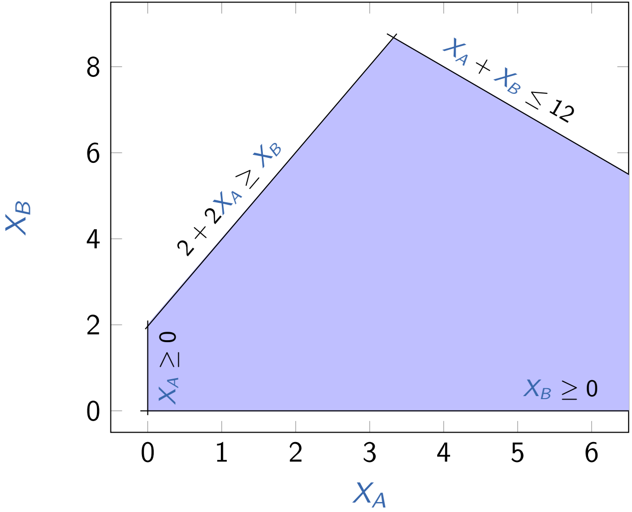
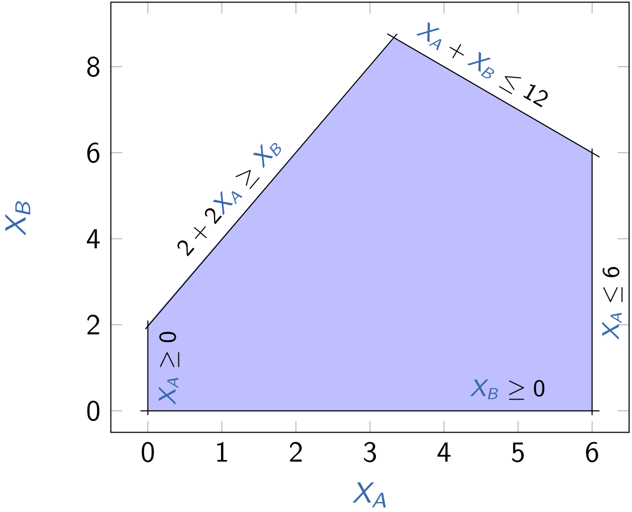


Integer Program Example: Visualization



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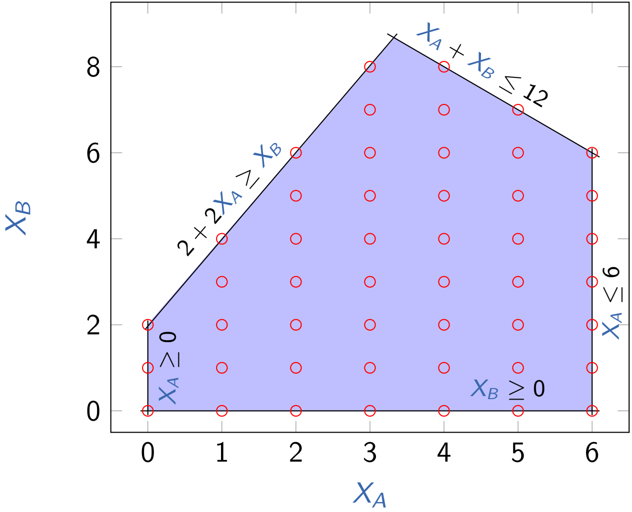

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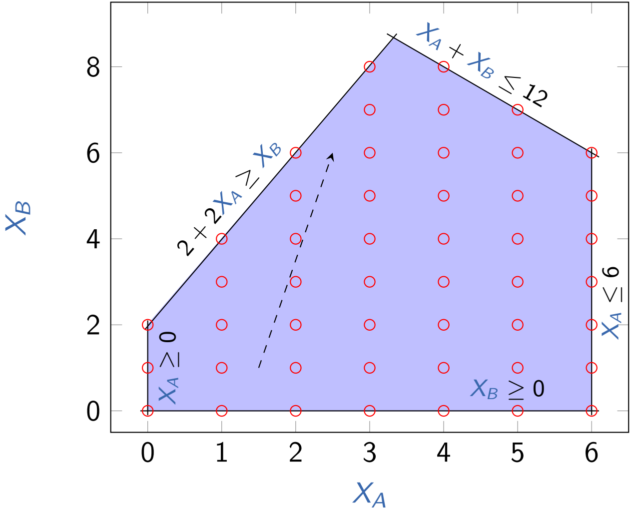

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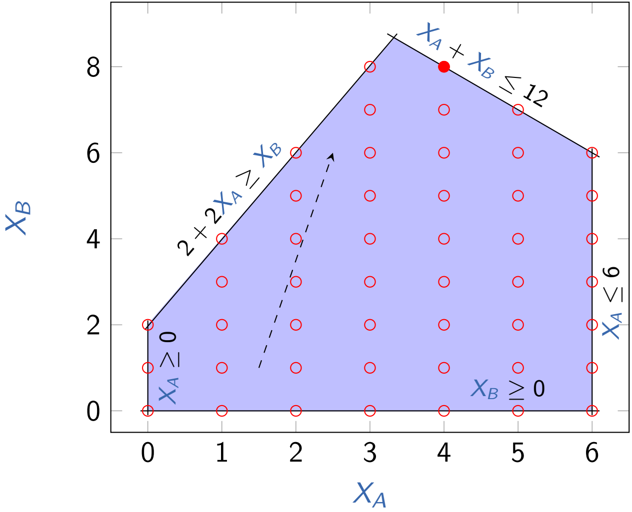


Integer Programs

Integer Program

An integer program (IP) consists of:

- a finite set of integer-valued variables V
- a finite set of linear inequalities (constraints) over V
- an objective function, which is a linear combination of V
- which should be minimized or maximized.

Terminology

- An integer assignment to all variables in V is feasible if it satisfies the constraints.
- An integer program is feasible if there is such a feasible assignment. Otherwise it is infeasible.
- A feasible maximum (resp. minimum) problem is unbounded if the objective function can assume arbitrarily large positive (resp. negative) values at feasible assignments. Otherwise it is bounded.
- The objective value of a bounded feasible maximum (resp. minimum) problem is the maximum (resp. minimum) value of the objective function with a feasible assignment.

Another Example

Example

$$\text{minimize} \quad 3X_{o_1} + 4X_{o_2} + 5X_{o_3} \quad \text{subject to}$$

$$\begin{aligned} X_{o_4} &\geq 1 \\ X_{o_1} + X_{o_2} &\geq 1 \\ X_{o_1} + X_{o_3} &\geq 1 \\ X_{o_2} + X_{o_3} &\geq 1 \end{aligned}$$

$$X_{o_1} \geq 0, \quad X_{o_2} \geq 0, \quad X_{o_3} \geq 0, \quad X_{o_4} \geq 0$$

What example from a recent chapter does this IP encode?

Another Example

Example

$$\text{minimize} \quad 3X_{o_1} + 4X_{o_2} + 5X_{o_3} \quad \text{subject to}$$

$$\begin{aligned} X_{o_4} &\geq 1 \\ X_{o_1} + X_{o_2} &\geq 1 \\ X_{o_1} + X_{o_3} &\geq 1 \\ X_{o_2} + X_{o_3} &\geq 1 \end{aligned}$$

$$X_{o_1} \geq 0, \quad X_{o_2} \geq 0, \quad X_{o_3} \geq 0, \quad X_{o_4} \geq 0$$

What example from a recent chapter does this IP encode?

$\rightsquigarrow$ the minimum hitting set from Chapter G4

Complexity of Solving Integer Programs

- As an IP can compute an MHS, solving an IP must be at least as complex as computing an MHS
- Reminder: MHS is a "classical" NP-complete problem
- Good news: Solving an IP is not harder

$\rightsquigarrow$ Finding solutions for IPs is NP-complete.

Complexity of Solving Integer Programs

- As an IP can compute an MHS, solving an IP must be at least as complex as computing an MHS
- Reminder: MHS is a "classical" NP-complete problem
- Good news: Solving an IP is not harder

$\rightsquigarrow$ Finding solutions for IPs is NP-complete.

Removing the requirement that solutions must be integer-valued leads to a simpler problem

Linear Programs

Linear Programs

Linear Program

A linear program (LP) consists of:

- a finite set of real-valued variables V
- a finite set of linear inequalities (constraints) over V
- an objective function, which is a linear combination of V
- which should be minimized or maximized.

We use the introduced IP terminology also for LPs.

Mixed IPs (MIPs) are something between IPs and LPs:
some variables are integer-valued, some are real-valued.

Linear Program: Example

Let X_A and X_B be the (real-valued) number of produced A and B

Example (Optimization Problem as Linear Program)

$$\text{maximize} \quad X_A + 5X_B \quad \text{subject to}$$

$$\begin{aligned} 2 + 2X_A &\geq X_B \\ X_A + X_B &\leq 12 \\ X_A &\leq 6 \end{aligned}$$

$$X_A \geq 0, \quad X_B \geq 0$$

Linear Program: Example

Let X_A and X_B be the (real-valued) number of produced A and B

Example (Optimization Problem as Linear Program)

maximize $\quad X_A + 5X_B \quad$ subject to

$$
\begin{aligned}
2 + 2X_A &\geq X_B \\
X_A + X_B &\leq 12 \\
X_A &\leq 6
\end{aligned}
$$

$$X_A \geq 0, \quad X_B \geq 0$$

$\rightsquigarrow$ unique optimal solution:
$X_A = 3\frac{1}{3}$ and $X_B = 8\frac{2}{3}$ with objective value $46\frac{2}{3}$

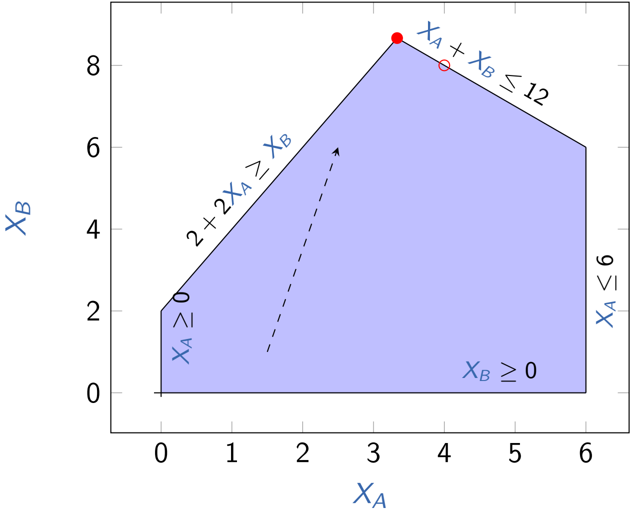
Same Program as Input for the CPLEX Solver

File `lp.lp`

```
Maximize
 obj: X_A + 5 X_B
Subject To
 c1: -2 X_A + X_B <= 2
 c2: X_A + X_B <= 12
Bounds
 0 <= X_A <= 6
 0 <= X_B
End
```

→ Demo

Linear Program Example: Visualization



Solving Linear Programs

- Observation:
  Here, LP solution is an upper bound for the corresponding IP.

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- Complexity:
  LP solving is a polynomial-time problem.

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- Observation:
  Here, LP solution is an upper bound for the corresponding IP.
- Complexity:
  LP solving is a polynomial-time problem.
- Common idea:
  Approximate IP solution with corresponding LP
  (LP relaxation).

LP Relaxation

Theorem (LP Relaxation)

The LP relaxation of an integer program is the problem that arises by removing the requirement that variables are integer-valued.

For a maximization (resp. minimization) problem, the objective value of the LP relaxation is an upper (resp. lower) bound on the value of the IP.

Proof idea.

Every feasible assignment for the IP is also feasible for the LP. □

LP Relaxation of MHS heuristic

Example (Minimum Hitting Set)

$$\text{minimize} \quad 3X_{o_1} + 4X_{o_2} + 5X_{o_3} \quad \text{subject to}$$

$$\begin{aligned} X_{o_4} &\geq 1 \\ X_{o_1} + X_{o_2} &\geq 1 \\ X_{o_1} + X_{o_3} &\geq 1 \\ X_{o_2} + X_{o_3} &\geq 1 \end{aligned}$$

$$X_{o_1} \geq 0, \quad X_{o_2} \geq 0, \quad X_{o_3} \geq 0, \quad X_{o_4} \geq 0$$

- ⇝ optimal solution of LP relaxation:
  $X_{o_4} = 1$ and $X_{o_1} = X_{o_2} = X_{o_3} = 0.5$ with objective value 6
- ⇝ LP relaxation of MHS heuristic is admissible
  and can be computed in polynomial time

Normal Forms and Duality

Standard Maximum Problem

Normal form for maximization problems:

Definition (Standard Maximum Problem)

Find values for $x_1, \ldots, x_n$, to maximize

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the constraints

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m
\end{aligned}$$

and $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$.

Standard Maximum Problem: Matrix and Vectors

A standard maximum problem is often given by

- an m-vector $\mathbf{b} = \langle b_1, \ldots, b_m \rangle^T$ (bounds),
- an n-vector $\mathbf{c} = \langle c_1, \ldots, c_n \rangle^T$ (objective coefficients),
- and an $m \times n$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \text{ (coefficients)}$$

- Then the problem is to find a vector $\mathbf{x} = \langle x_1, \ldots, x_n \rangle^T$ to maximize $\mathbf{c}^T\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Standard Minimum Problem

- there is also a standard minimum problem
- it's form is identical to the standard maximum problem, except that
  - the aim is to minimize the objective function
  - subject to $\mathbf{A}\mathbf{x} \geq \mathbf{b}$
- All linear programs can efficiently be converted into a standard maximum/minimum problem.

Some LP Theory: Duality

Every LP has an alternative view (its dual LP).

Primal	Dual
maximization (or minimization)	minimization (or maximization)
objective coefficients	bounds
bounds	objective coefficients
bounded variable	$\geq$-constraint
$\leq$-constraint	bounded variable
free variable	$=$-constraint
$=$-constraint	free variable

dual of dual: original LP

Dual Problem

Definition (Dual Problem)

The dual of the standard maximum problem

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } \mathbf{A}\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}$$

is the standard minimum problem

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0}$$

Dual Problem: Example

Example (Dual of the Optimization Problem)

$$\text{maximize} \quad X_A + 5X_B \quad \text{subject to}$$

$$\begin{aligned} -2X_A + X_B &\leq 2 \\ X_A + X_B &\leq 12 \\ X_A &\leq 6 \end{aligned}$$

$$X_A \geq 0, \quad X_B \geq 0$$

Dual Problem: Example

Example (Dual of the Optimization Problem)

$$\text{maximize} \quad X_A + 5X_B \quad \text{subject to}$$

$$
\begin{aligned}
&[Y_1] & -2X_A + X_B &\leq 2 \\
&[Y_2] & X_A + X_B &\leq 12 \\
&[Y_3] & X_A &\leq 6
\end{aligned}
$$

$$X_A \geq 0, \quad X_B \geq 0$$

Dual Problem: Example

Example (Dual of the Optimization Problem)

minimize $2Y_1 + 12Y_2 + 6Y_3$ subject to

$$[X_A] \quad -2Y_1 + Y_2 + Y_3 \geq 1$$
$$[X_B] \quad Y_1 + Y_2 \geq 5$$

$$Y_1 \geq 0, \quad Y_2 \geq 0, \; Y_3 \geq 0$$

Duality Theorem

Theorem (Duality Theorem)

If a standard linear program is bounded feasible, then so is its dual, and their objective values are equal.

(Proof omitted.)

The dual provides a different perspective on a problem.

Summary

Summary

- Linear (and integer) programs consist of an objective function that should be maximized or minimized subject to a set of given linear constraints.
- Finding solutions for integer programs is NP-complete.
- LP solving is a polynomial time problem.
- The dual of a maximization LP is a minimization LP and vice versa.
- The dual of a bounded feasible LP has the same objective value.

Further Reading

The slides in this chapter are based on the following excellent tutorial on LP solving:

Thomas S. Ferguson.
Linear Programming – A Concise Introduction.
UCLA, unpublished document available online.