Planning and Optimization

E5. Abstractions: Additive Abstractions

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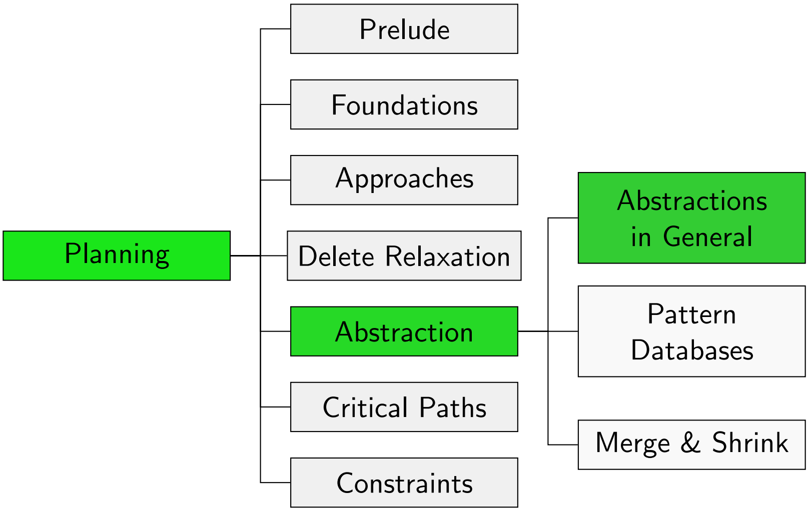
Content of this Course



Additivity

Orthogonality of Abstractions

Definition (Orthogonal)

Let α_1 and α_2 be abstractions of transition system $\mathcal{T}$.

We say that α_1 and α_2 are orthogonal if for all transitions $s \xrightarrow{\ell} t$ of $\mathcal{T}$, we have $\alpha_1(s) = \alpha_1(t)$ or $\alpha_2(s) = \alpha_2(t)$.

Affecting Transition Labels

Definition (Affecting Transition Labels)

Let $\mathcal{T}$ be a transition system, and let ℓ be one of its labels.

We say that ℓ affects $\mathcal{T}$ if $\mathcal{T}$ has a transition $s \xrightarrow{\ell} t$ with $s \neq t$.

Theorem (Affecting Labels vs. Orthogonality)

Let α_1 and α_2 be abstractions of transition system $\mathcal{T}$.

If no label of $\mathcal{T}$ affects both $\mathcal{T}^{\alpha_1}$ and $\mathcal{T}^{\alpha_2}$,
then α_1 and α_2 are orthogonal.

(Easy proof omitted.)

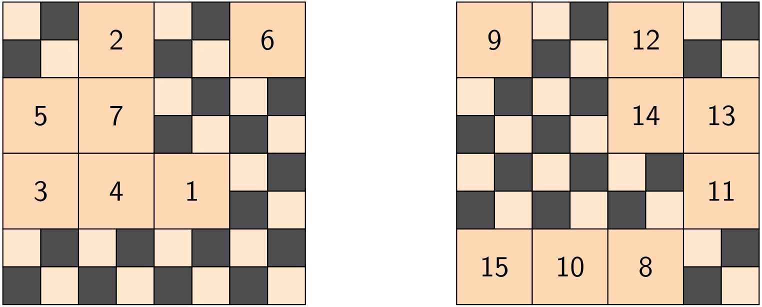
Orthogonal Abstractions: Example

	2		6
5	7		
3	4	1	

9		12	
		14	13
			11
15	10	8	

Are the abstractions orthogonal?

Orthogonal Abstractions: Example



Are the abstractions orthogonal?

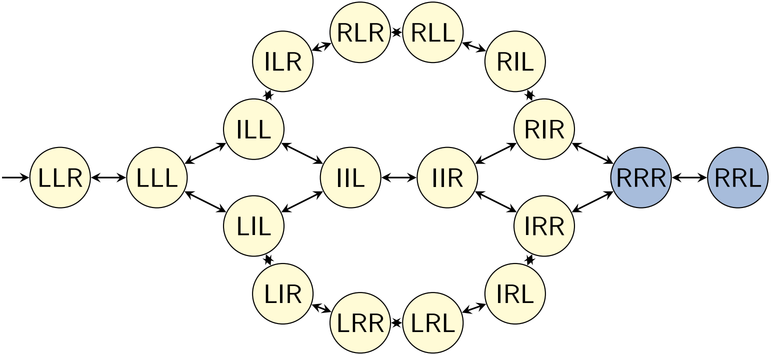
Orthogonality and Additivity

Theorem (Additivity for Orthogonal Abstractions)

Let $h^{\alpha_1}, \dots, h^{\alpha_n}$ be abstraction heuristics of the same transition system such that α_i and α_j are orthogonal for all $i \neq j$.

Then $\sum_{i=1}^{n} h^{\alpha_i}$ is a safe, goal-aware, admissible and consistent heuristic for Π.

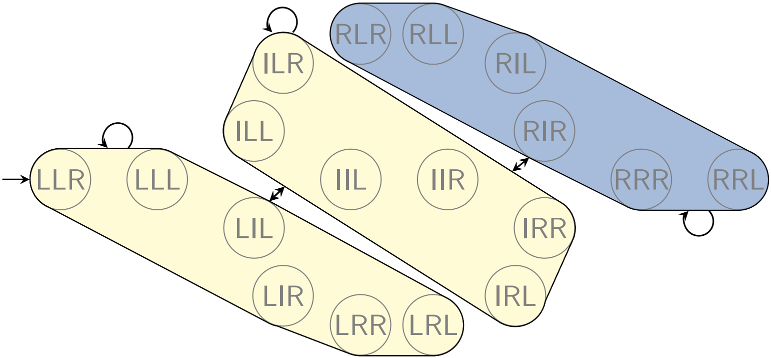
Orthogonality and Additivity: Example



transition system $\mathcal{T}$

state variables: first package, second package, truck

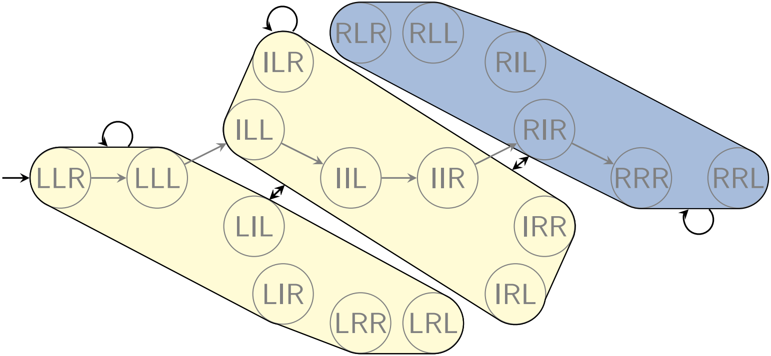
Orthogonality and Additivity: Example



abstraction α_1

abstraction: only consider value of first package

Orthogonality and Additivity: Example



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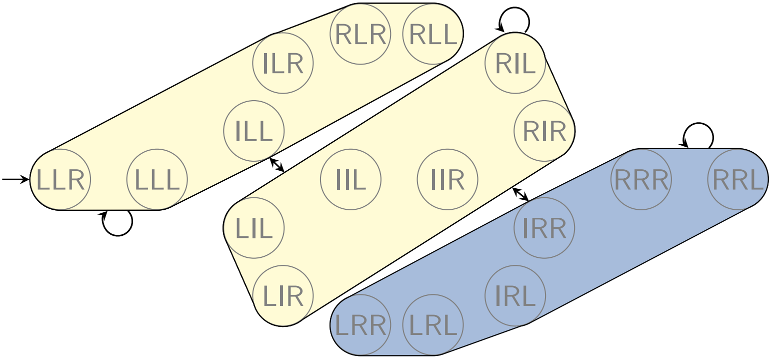
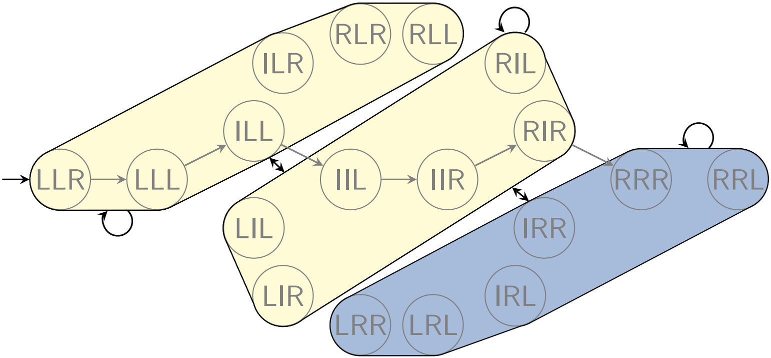
Orthogonality and Additivity: Example



abstraction α_2 (orthogonal to α_1)

abstraction: only consider value of second package

Orthogonality and Additivity: Example



abstraction α_2 (orthogonal to α_1)

abstraction: only consider value of second package

Orthogonality and Additivity: Proof (1)

Proof.

We prove goal-awareness and consistency;
the other properties follow from these two.

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ be the concrete transition system.

Let $h = \sum_{i=1}^{n} h^{\alpha_i}$.

Orthogonality and Additivity: Proof (1)

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Let $h = \sum_{i=1}^{n} h^{\alpha_i}$.

Goal-awareness: For goal states $s \in S_\star$,
$h(s) = \sum_{i=1}^{n} h^{\alpha_i}(s) = \sum_{i=1}^{n} 0 = 0$ because all individual abstraction heuristics are goal-aware. ...

Orthogonality and Additivity: Proof (2)

Proof (continued).

Consistency: Let $s \xrightarrow{o} t \in T$. We must prove $h(s) \leq c(o) + h(t)$.

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Because the abstractions are orthogonal, $\alpha_i(s) \neq \alpha_i(t)$
for at most one $i \in \{1, \dots, n\}$.

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Case 1: $\alpha_i(s) = \alpha_i(t)$ for all $i \in \{1, \dots, n\}$.

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Proof (continued).

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Case 1: $\alpha_i(s) = \alpha_i(t)$ for all $i \in \{1, \ldots, n\}$.

Then
$$
\begin{aligned}
h(s) &= \sum_{i=1}^{n} h^{\alpha_i}(s) \\
&= \sum_{i=1}^{n} h^*_{\mathcal{T}^{\alpha_i}}(\alpha_i(s)) \\
&= \sum_{i=1}^{n} h^*_{\mathcal{T}^{\alpha_i}}(\alpha_i(t)) \\
&= \sum_{i=1}^{n} h^{\alpha_i}(t) \\
&= h(t) \leq c(o) + h(t).
\end{aligned}
$$

...

Orthogonality and Additivity: Proof (3)

Proof (continued).

Case 2: $\alpha_i(s) \neq \alpha_i(t)$ for exactly one $i \in \{1, \ldots, n\}$.

Let $k \in \{1, \ldots, n\}$ such that $\alpha_k(s) \neq \alpha_k(t)$.

Orthogonality and Additivity: Proof (3)

Proof (continued).

Case 2: $\alpha_i(s) \neq \alpha_i(t)$ for exactly one $i \in \{1, \dots, n\}$.

Let $k \in \{1, \dots, n\}$ such that $\alpha_k(s) \neq \alpha_k(t)$.

Then
$$\begin{aligned} h(s) &= \textstyle\sum_{i=1}^{n} h^{\alpha_i}(s) \\ &= \textstyle\sum_{i \in \{1,\dots,n\} \setminus \{k\}} h^*_{\mathcal{T}^{\alpha_i}}(\alpha_i(s)) + h^{\alpha_k}(s) \\ &\leq \textstyle\sum_{i \in \{1,\dots,n\} \setminus \{k\}} h^*_{\mathcal{T}^{\alpha_i}}(\alpha_i(t)) + c(o) + h^{\alpha_k}(t) \\ &= c(o) + \textstyle\sum_{i=1}^{n} h^{\alpha_i}(t) \\ &= c(o) + h(t), \end{aligned}$$
where the inequality holds because $\alpha_i(s) = \alpha_i(t)$ for all $i \neq k$ and h^{α_k} is consistent. $\square$

Outlook

Using Abstraction Heuristics in Practice

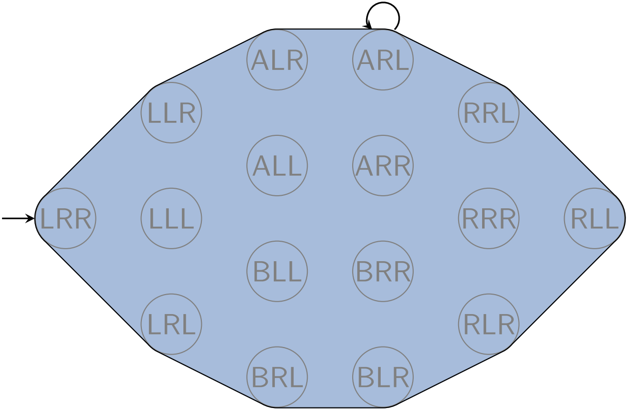
In practice, there are conflicting goals for abstractions:

- we want to obtain an informative heuristic, but
- want to keep its representation small.

Abstractions have small representations if

- there are few abstract states and
- there is a succinct encoding for α.

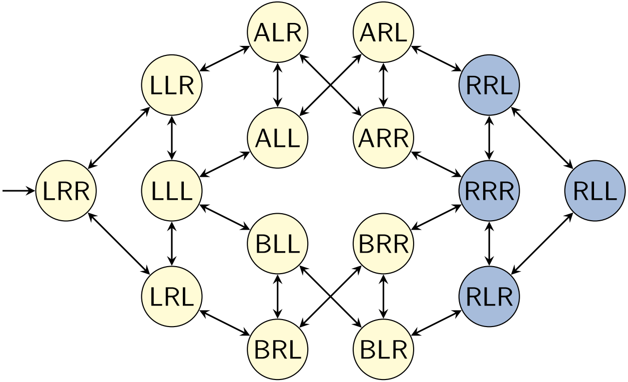
Counterexample: One-State Abstraction



One-state abstraction: $\alpha(s) :=$ const.

- $+$ very few abstract states and succinct encoding for α
- $-$ completely uninformative heuristic

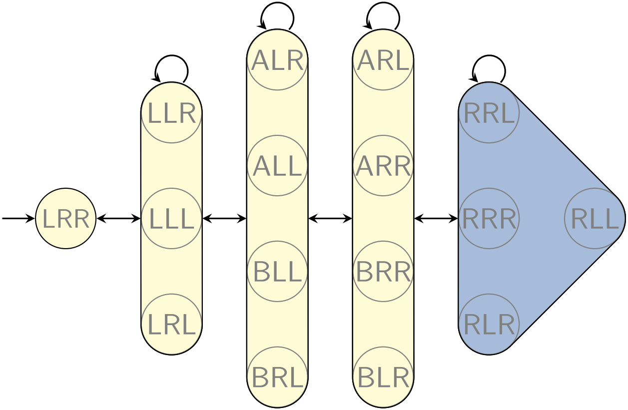
Counterexample: Identity Abstraction



Identity abstraction: $\alpha(s) := s$.

+ perfect heuristic and succinct encoding for α
− too many abstract states

Counterexample: Perfect Abstraction



Perfect abstraction: $\alpha(s) := h^*(s)$.

+ perfect heuristic and usually few abstract states
− usually no succinct encoding for α

Automatically Deriving Good Abstraction Heuristics

Abstraction Heuristics for Planning: Main Research Problem

Automatically derive effective abstraction heuristics
for planning tasks.

$\rightsquigarrow$ we will study two state-of-the-art approaches
in the following chapters

Summary

Summary

- Abstraction heuristics from orthogonal abstractions can be added without losing admissibility or consistency.
- One sufficient condition for orthogonality is that all abstractions are affected by disjoint sets of labels.
- Practically useful abstractions are those which give informative heuristics, yet have a small representation.
- Coming up with good abstractions automatically is the main research challenge when applying abstraction heuristics in planning.