Planning and Optimization

C6. Symbolic Search: Binary Decision Diagrams

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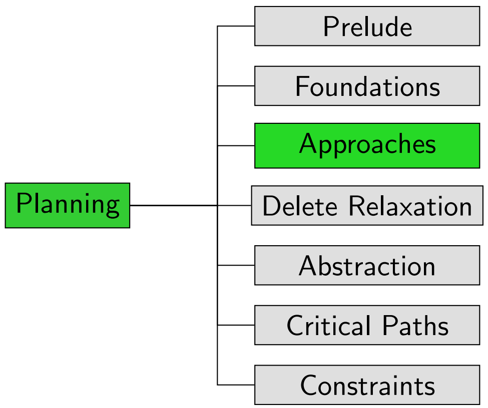


C6.1 Motivation

Symbolic Search Planning: Basic Ideas

- come up with a good data structure for sets of states
- hope: (at least some) exponentially large state sets can be represented as polynomial-size data structures
- simulate a standard search algorithm like breadth-first search using these set representations

Symbolic Breadth-First Progression Search

Symbolic Breadth-First Progression Search

def bfs-progression(V, I, O, γ):
  $goal\_states := models(\gamma)$
  $reached_0 := \{I\}$
  $i := 0$
  loop:
    if $reached_i \cap goal\_states \neq \emptyset$:
      return solution found
    $reached_{i+1} := reached_i \cup apply(reached_i, O)$
    if $reached_{i+1} = reached_i$:
      return no solution exists
    $i := i + 1$

$\rightsquigarrow$ If we can implement operations *models*, $\{I\}$, $\cap$, $\neq \emptyset$, $\cup$, *apply* and $=$ efficiently, this is a reasonable algorithm.

C6.2 Data Structures for State Sets

Representing State Sets

We need to represent and manipulate state sets (again)!

- How about an explicit representation, like a hash table?
- And how about our good old friend, the formula?

Time Complexity: Explicit Representations vs. Formulas

Let k be the number of state variables,
$|S|$ the number of states in S and
$\|S\|$ the size of the representation of S.

	Hash table	Formula
$s \in S$?	$O(k)$	$O(\|S\|)$
$S := S \cup \{s\}$	$O(k)$	$O(k)$
$S := S \setminus \{s\}$	$O(k)$	$O(k)$
$S \cup S'$	$O(k\lvert S\rvert + k\lvert S'\rvert)$	$O(1)$
$S \cap S'$	$O(k\lvert S\rvert + k\lvert S'\rvert)$	$O(1)$
$S \setminus S'$	$O(k\lvert S\rvert + k\lvert S'\rvert)$	$O(1)$
$\overline{S}$	$O(k2^k)$	$O(1)$
$\{s \mid s(v) = \mathbf{T}\}$	$O(k2^k)$	$O(1)$
$S = \emptyset$?	$O(1)$	co-NP-complete
$S = S'$?	$O(k\lvert S\rvert)$	co-NP-complete
$\lvert S\rvert$	$O(1)$	#P-complete

Which Operations are Important?

- Explicit representations such as hash tables are unsuitable because their size grows linearly with the number of represented states.
- Formulas are very efficient for some operations, but not for other important operations needed by the breadth-first search algorithm.
  - Examples: $S \neq \emptyset$?, $S = S'$?

Canonical Representations

- One of the problems with formulas is that they allow many different representations for the same set.
  - For example, all unsatisfiable formulas represent $\emptyset$.

  This makes equality tests expensive.
- We would like data structures with a canonical representation, i.e., with only one possible representation for every state set.
- Reduced ordered binary decision diagrams (BDDs) are an example of such a canonical representation.

Time Complexity: Formulas vs. BDDs

Let k be the number of state variables,
$|S|$ the number of states in S and
$\|S\|$ the size of the representation of S.

	Formula	BDD
$s \in S$?	$O(\|S\|)$	$O(k)$
$S := S \cup \{s\}$	$O(k)$	$O(k)$
$S := S \setminus \{s\}$	$O(k)$	$O(k)$
$S \cup S'$	$O(1)$	$O(\|S\|\|S'\|)$
$S \cap S'$	$O(1)$	$O(\|S\|\|S'\|)$
$S \setminus S'$	$O(1)$	$O(\|S\|\|S'\|)$
$\overline{S}$	$O(1)$	$O(\|S\|)$
$\{s \mid s(v) = \mathbf{T}\}$	$O(1)$	$O(1)$
$S = \emptyset$?	co-NP-complete	$O(1)$
$S = S'$?	co-NP-complete	$O(1)$
$\lvert S\rvert$	#P-complete	$O(\|S\|)$

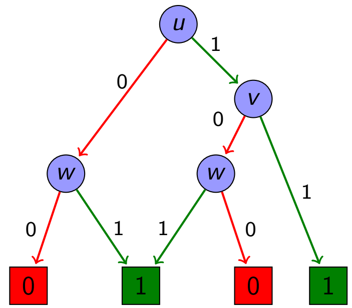
Remark: Optimizations allow BDDs with complementation ($\overline{S}$) in constant time, but we will not discuss this here.

C6.3 Binary Decision Diagrams

BDD Example

Example

Possible BDD for $(u \wedge v) \vee w$



Binary Decision Diagrams: Definition

Definition (BDD)

Let V be a set of propositional variables.

A binary decision diagram (BDD) over V is a directed acyclic graph with labeled arcs and labeled vertices such that:

- There is exactly one node without incoming arcs.
- All sinks (nodes without outgoing arcs) are labeled 0 or 1.
- All other nodes are labeled with a variable $v \in V$ and have exactly two outgoing arcs, labeled 0 and 1.

A note on notation:

- In BDDs, 1 stands for **T** and 0 for **F**.
- We follow this customary notation in BDDs, but stick to **T** and **F** when speaking of logic.

Binary Decision Diagrams: Terminology

BDD Terminology

- The node without incoming arcs is called the root.
- The labeling variable of an internal node is called the decision variable of the node.
- The nodes reached from node n via the arc labeled $i \in \{0, 1\}$ is called the i-successor of n.
- The BDDs which only consist of a single sink are called the zero BDD and one BDD.

Observation: If B is a BDD and n is a node of B, then the subgraph induced by all nodes reachable from n is also a BDD.

- This BDD is called the BDD rooted at n.

BDD Semantics

Testing whether a BDD Includes a Variable Assignment

def bdd-includes(B: BDD, I: variable assignment):
    Set n to the root of B.
    while n is not a sink:
        Set v to the decision variable of n.
        Set n to the 1-successor of n if $I(v) = \mathbf{T}$ and
          to the 0-successor of n if $I(v) = \mathbf{F}$.
    return true if n is labeled 1, **false** if it is labeled 0.

Definition (Set Represented by a BDD)

Let B be a BDD over variables V.

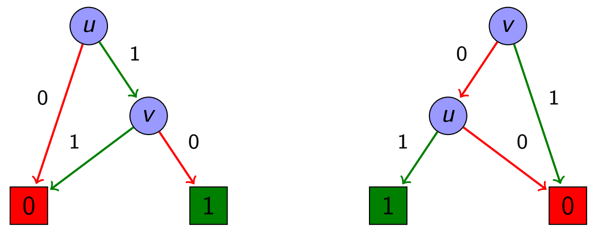
The set represented by B, in symbols $r(B)$, consists of all variable assignments $I : V \to \{\mathbf{T}, \mathbf{F}\}$ for which *bdd-includes*(B, I) returns true.

C6.4 BDDs as Canonical Representations

Ordered BDDs: Motivation

In general, BDDs are not a canonical representation for sets of interpretations. Here is a simple counter-example ($V = \{u, v\}$):

Example (BDDs for $u \land \neg v$ with Different Variable Order)



Both BDDs represent the same state set, namely the singleton set $\{\{u \mapsto \mathbf{T}, v \mapsto \mathbf{F}\}\}$.

Ordered BDDs: Definition

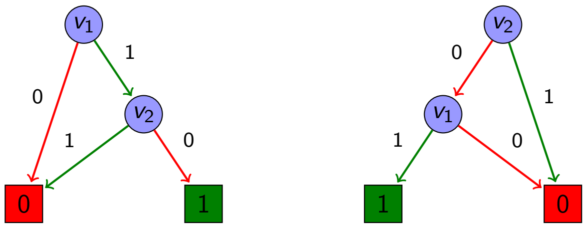
- As a first step towards a canonical representation, we now require that the set of variables is totally ordered by some ordering $\prec$.
- In particular, we will only use variables $v_1, v_2, v_3, \dots$ and assume the ordering $v_i \prec v_j$ iff $i < j$.

Definition (Ordered BDD)

A BDD is ordered (w.r.t. $\prec$) iff for each arc from a node with decision variable u to a node with decision variable v, we have $u \prec v$.

Ordered BDDs: Example

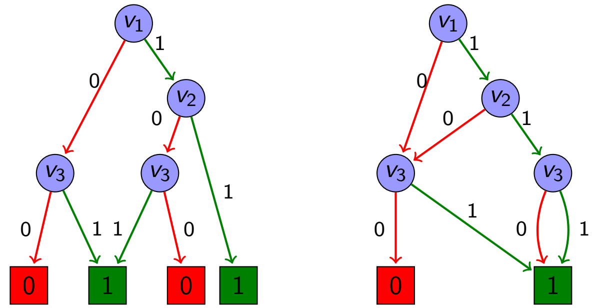
Example (Ordered and Unordered BDD)



The left BDD is ordered w.r.t. the ordering we use in this chapter, the right one is not.

Reduced Ordered BDDs: Are Ordered BDDs Canonical?

Example (Two equivalent BDDs that can be reduced)



- Ordered BDDs are still not canonical:
  both ordered BDDs represent the same set.
- However, ordered BDDs can easily be made canonical.

Reduced Ordered BDDs: Reductions (1)

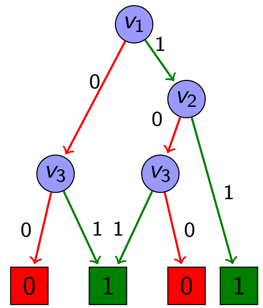
There are two important operations on BDDs
that do not change the set represented by it:

Definition (Isomorphism Reduction)

If the BDDs rooted at two different nodes n and n' are isomorphic,
then all incoming arcs of n' can be redirected to n,
and all BDD nodes unreachable from the root can be removed.

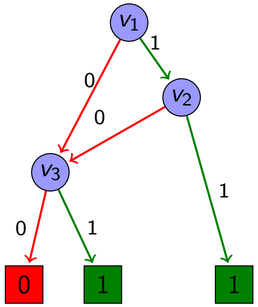
Reduced Ordered BDDs: Reductions (2)

Example (Isomorphism Reduction)



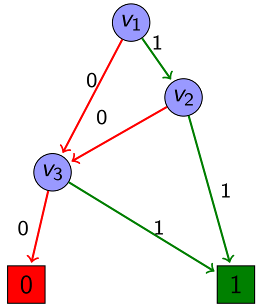
Reduced Ordered BDDs: Reductions (2)

Example (Isomorphism Reduction)



Reduced Ordered BDDs: Reductions (2)

Example (Isomorphism Reduction)



Reduced Ordered BDDs: Reductions (3)

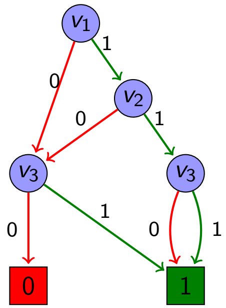
There are two important operations on BDDs that do not change the set represented by it:

Definition (Shannon Reduction)

If both outgoing arcs of an internal node n of a BDD lead to the same node m, then n can be removed from the BDD, with all incoming arcs of n going to m instead.

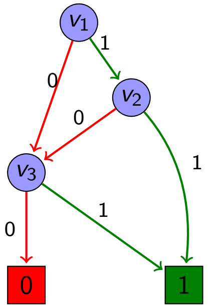
Reduced Ordered BDDs: Reductions (4)

Example (Shannon Reduction)



Reduced Ordered BDDs: Reductions (4)

Example (Shannon Reduction)



Reduced Ordered BDDs: Definition

Definition (Reduced Ordered BDD)

An ordered BDD is reduced iff it does not admit any isomorphism reduction or Shannon reduction.

Theorem (Bryant 1986)

For every state set S and a fixed variable ordering, there exists exactly one reduced ordered BDD representing S.

Moreover, given any ordered BDD B, the equivalent reduced ordered BDD can be computed in linear time in the size of B.

$\rightsquigarrow$ Reduced ordered BDDs are the canonical representation we are looking for.

From now on, we simply say BDD for reduced ordered BDD.

C6.5 Summary

Summary

- Symbolic search is based on the idea of performing a state-space search where many states are considered "at once" by operating on sets of states rather than individual states.
- Binary decision diagrams are a data structure to compactly represent and manipulate sets of variable assignments.
- Reduced ordered BDDs are a canonical representation of such sets.