Planning and Optimization

B6. Computational Complexity of Planning

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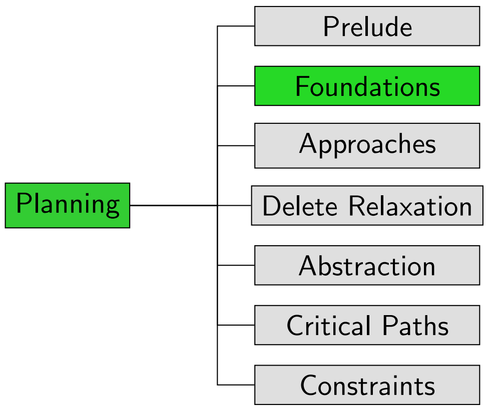
Content of this Course



B6.1 Motivation

How Difficult is Planning?

- Using state-space search (e.g., using Dijkstra's algorithm on the transition system), planning can be solved in polynomial time in the number of states.
- However, the number of states is exponential in the number of state variables, and hence in general exponential in the size of the input to the planning algorithm.

⇝ Do non-exponential planning algorithms exist?

⇝ What is the precise computational complexity of planning?

Why Computational Complexity?

- understand the problem
- know what is not possible
- find interesting subproblems that are easier to solve
- distinguish essential features from syntactic sugar
  - Is STRIPS planning easier than general planning?

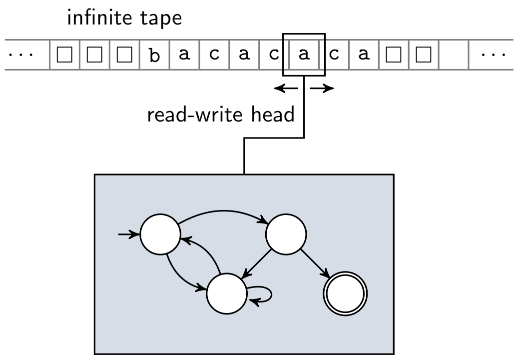
B6.2 Background: Complexity Theory

Reminder: Complexity Theory

Need to Catch Up?

- We assume knowledge of complexity theory:
  - languages and decision problems
  - Turing machines: NTMs and DTMs; polynomial equivalence with other models of computation
  - complexity classes: P, NP, PSPACE
  - polynomial reductions
- If you are not familiar with these topics, we recommend Chapters B10, D1–D3, D6 of the Theory of Computer Science course at `https://dmi.unibas.ch/de/studium/computer-science-informatik/lehrangebot-fs23/main-lecture-theory-of-computer-science-1/`

Turing Machines: Conceptually



Turing Machines

Definition (Nondeterministic Turing Machine)

A nondeterministic Turing machine (NTM) is a 6-tuple $\langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$ with the following components:

- input alphabet Σ and blank symbol $\square \notin \Sigma$
  - alphabets always nonempty and finite
  - tape alphabet $\Sigma_\square = \Sigma \cup \{\square\}$
- finite set Q of internal states with initial state $q_0 \in Q$ and accepting state $q_Y \in Q$
  - nonterminal states $Q' := Q \setminus \{q_Y\}$
- transition relation $\delta : (Q' \times \Sigma_\square) \to 2^{Q \times \Sigma_\square \times \{-1,+1\}}$

Deterministic Turing machine (DTM):
$|\delta(q, s)| = 1$ for all $\langle q, s \rangle \in Q' \times \Sigma_\square$

Turing Machines: Accepted Words

- Initial configuration
  - state q_0
  - input word on tape, all other tape cells contain $\square$
  - head on first symbol of input word
- Step
  - If in state q, reading symbol s, and $\langle q', s', d \rangle \in \delta(q, s)$ then
  - the NTM can transition to state q', replacing s with s' and moving the head one cell to the left/right ($d = -1/+1$).
- Input word ($\in \Sigma^*$) is accepted if some sequence of transitions brings the NTM from the initial configuration into state s_Y.

Acceptance in Time and Space

Definition (Acceptance of a Language in Time/Space)

Let $f : \mathbb{N}_0 \to \mathbb{N}_0$.

A NTM accepts language $L \subseteq \Sigma^*$ in time f if it accepts each $w \in L$ within $f(|w|)$ steps and does not accept any $w \notin L$ (in any time).

It accepts language $L \subseteq \Sigma^*$ in space f if it accepts each $w \in L$ using at most $f(|w|)$ tape cells and does not accept any $w \notin L$.

Time and Space Complexity Classes

Definition (DTIME, NTIME, DSPACE, NSPACE)

Let $f : \mathbb{N}_0 \to \mathbb{N}_0$.

Complexity class DTIME(f) contains all languages accepted in time f by some DTM.

Complexity class NTIME(f) contains all languages accepted in time f by some NTM.

Complexity class DSPACE(f) contains all languages accepted in space f by some DTM.

Complexity class NSPACE(f) contains all languages accepted in space f by some NTM.

Polynomial Time and Space Classes

Let $\mathcal{P}$ be the set of polynomials $p : \mathbb{N}_0 \to \mathbb{N}_0$ whose coefficients are natural numbers.

Definition (P, NP, PSPACE, NPSPACE)

$$\begin{aligned} \mathrm{P} &= \textstyle\bigcup_{p\in\mathcal{P}} \mathrm{DTIME}(p) \\ \mathrm{NP} &= \textstyle\bigcup_{p\in\mathcal{P}} \mathrm{NTIME}(p) \\ \mathrm{PSPACE} &= \textstyle\bigcup_{p\in\mathcal{P}} \mathrm{DSPACE}(p) \\ \mathrm{NPSPACE} &= \textstyle\bigcup_{p\in\mathcal{P}} \mathrm{NSPACE}(p) \end{aligned}$$

Polynomial Complexity Class Relationships

Theorem (Complexity Class Hierarchy)

$\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} = \mathrm{NPSPACE}$

Proof.

$\mathrm{P} \subseteq \mathrm{NP}$ and $\mathrm{PSPACE} \subseteq \mathrm{NPSPACE}$ are obvious because deterministic Turing machines are a special case of nondeterministic ones.

$\mathrm{NP} \subseteq \mathrm{NPSPACE}$ holds because a Turing machine can only visit polynomially many tape cells within polynomial time.

$\mathrm{PSPACE} = \mathrm{NPSPACE}$ is a special case of a classical result known as Savitch's theorem (Savitch 1970). □

B6.3 (Bounded-Cost) Plan Existence

Decision Problems for Planning

Definition (Plan Existence)

Plan existence (PlanEx) is the following decision problem:

Given: planning task Π
Question: Is there a plan for Π?

$\rightsquigarrow$ decision problem analogue of satisficing planning

Definition (Bounded-Cost Plan Existence)

Bounded-cost plan existence (BCPlanEx) is the following decision problem:

Given: planning task Π, cost bound $K \in \mathbb{N}_0$
Question: Is there a plan for Π with cost at most K?

$\rightsquigarrow$ decision problem analogue of optimal planning

Plan Existence vs. Bounded-Cost Plan Existence

Theorem (Reduction from PlanEx to BCPlanEx)

PlanEx $\leq_p$ BCPlanEx

Proof.

Consider a planning task Π with state variables V.

Let $c_{\max}$ be the maximal cost of all operators of Π.

Compute the number of states of Π as $N = 2^{|V|}$.

Π is solvable iff there is solution with cost at most $c_{\max} \cdot (N-1)$ because a solution need not visit any state twice.

$\rightsquigarrow$ map instance Π of PlanEx to instance $\langle \Pi, c_{\max} \cdot (N-1) \rangle$ of BCPlanEx

$\rightsquigarrow$ polynomial reduction □

B6.4 PSPACE-Completeness of Planning

Membership in PSPACE

Theorem

BCPlanEx $\in$ PSPACE

Proof.

Show BCPlanEx $\in$ NPSPACE and use Savitch's theorem.

Nondeterministic algorithm:

def plan($\langle V, I, O, \gamma \rangle$, K):
    $s := I$
    $k := K$
    loop forever:
        if $s \models \gamma$: **accept**
        guess $o \in O$
        if o is not applicable in s: **fail**
        if $cost(o) > k$: **fail**
        $s := s[\![o]\!]$
        $k := k - cost(o)$

□

PSPACE-Hardness

Idea: generic reduction

- For an arbitrary fixed DTM M with space bound polynomial p and input w, generate propositional planning task which is solvable iff M accepts w in space $p(|w|)$.
- Without loss of generality, we assume $p(n) \geq n$ for all n.

Reduction: State Variables

Let $M = \langle \Sigma, \Box, Q, q_0, q_Y, \delta \rangle$ be the fixed DTM, and let p be its space-bound polynomial.

Given input $w_1 \dots w_n$, define relevant tape positions $X := \{-p(n), \dots, p(n)\}$

State Variables

- state_q for all $q \in Q$
- head_i for all $i \in X \cup \{-p(n) - 1, p(n) + 1\}$
- $\mathrm{content}_{i,a}$ for all $i \in X$, $a \in \Sigma_\Box$

$\rightsquigarrow$ allows encoding a Turing machine configuration

Reduction: Initial State

Let $M = \langle \Sigma, \Box, Q, q_0, q_Y, \delta \rangle$ be the fixed DTM, and let p be its space-bound polynomial.

Given input $w_1 \dots w_n$, define relevant tape positions $X := \{-p(n), \dots, p(n)\}$

Initial State

Initially true:

- state_{q_0}
- head_1
- $\mathrm{content}_{i,w_i}$ for all $i \in \{1, \dots, n\}$
- $\mathrm{content}_{i,\Box}$ for all $i \in X \setminus \{1, \dots, n\}$

Initially false:

- all others

Reduction: Operators

Let $M = \langle \Sigma, \Box, Q, q_0, q_Y, \delta \rangle$ be the fixed DTM, and let p be its space-bound polynomial.

Given input $w_1 \dots w_n$, define relevant tape positions $X := \{-p(n), \dots, p(n)\}$

Operators

One operator for each transition rule $\delta(q, a) = \langle q', a', d \rangle$ and each cell position $i \in X$:

- precondition: $\mathrm{state}_q \land \mathrm{head}_i \land \mathrm{content}_{i,a}$
- effect: $\neg\mathrm{state}_q \land \neg\mathrm{head}_i \land \neg\mathrm{content}_{i,a}$
  $\land\, \mathrm{state}_{q'} \land \mathrm{head}_{i+d} \land \mathrm{content}_{i,a'}$

Note that add-after-delete semantics are important here!

Reduction: Goal

Let $M = \langle \Sigma, \Box, Q, q_0, q_Y, \delta \rangle$ be the fixed DTM, and let p be its space-bound polynomial.

Given input $w_1 \dots w_n$, define relevant tape positions $X := \{-p(n), \dots, p(n)\}$

Goal

state_{q_Y}

PSPACE-Completeness of STRIPS Plan Existence

Theorem (PSPACE-Completeness; Bylander, 1994)

PlanEx and BCPlanEx are PSPACE-complete. This is true even if only STRIPS tasks are allowed.

Proof.

Membership for BCPlanEx was already shown.

Hardness for PlanEx follows because we just presented a polynomial reduction from an arbitrary problem in PSPACE to PlanEx. (Note that the reduction only generates STRIPS tasks, after trivial cleanup to make them conflict-free.)

Membership for PlanEx and hardness for BCPlanEx follow from the polynomial reduction from PlanEx to BCPlanEx. □

B6.5 More Complexity Results

More Complexity Results

In addition to the basic complexity result presented in this chapter, there are many special cases, generalizations, variations and related problems studied in the literature:

- different planning formalisms
  - e.g., nondeterministic effects, partial observability, schematic operators, numerical state variables
- syntactic restrictions of planning tasks
  - e.g., without preconditions, without conjunctive effects, STRIPS without delete effects
- semantic restrictions of planning task
  - e.g., restricting variable dependencies ("causal graphs")
- particular planning domains
  - e.g., Blocksworld, Logistics, FreeCell

Complexity Results for Different Planning Formalisms

Some results for different planning formalisms:

- nondeterministic effects:
  - fully observable: EXP-complete (Littman, 1997)
  - unobservable: EXPSPACE-complete (Haslum & Jonsson, 1999)
  - partially observable: 2-EXP-complete (Rintanen, 2004)
- schematic operators:
  - usually adds one exponential level to PLANEX complexity
  - e.g., classical case EXPSPACE-complete (Erol et al., 1995)
- numerical state variables:
  - undecidable in most variations (Helmert, 2002)

B6.6 Summary

Summary

- PSPACE: decision problems solvable in polynomial space
- P $\subseteq$ NP $\subseteq$ PSPACE $=$ NPSPACE.
- Classical planning is PSPACE-complete.
- This is true both for satisficing and optimal planning (rather, the corresponding decision problems).
- The hardness proof is a polynomial reduction that translates an arbitrary polynomial-space DTM into a STRIPS task:
  - DTM configurations are encoded by state variables.
  - Operators simulate transitions between DTM configurations.
  - The DTM accepts an input iff there is a plan for the corresponding STRIPS task.
- This implies that there is no polynomial algorithm for classical planning unless P $=$ PSPACE.
- It also means that planning is not polynomially reducible to any problem in NP unless NP $=$ PSPACE.