

# Discrete Mathematics in Computer Science

## D6. Advanced Concepts in Predicate Logic and Outlook

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# Free and Bound Variables

# Free and Bound Variables: Motivation

## Question:

- Consider a signature with variable symbols  $\{x_1, x_2, x_3, \dots\}$  and an interpretation  $\mathcal{I}$ .
- Which parts of the definition of  $\alpha$  are relevant to decide whether  $\mathcal{I}, \alpha \models (\forall x_4 (R(x_4, x_2) \vee (f(x_3) = x_4))) \vee \exists x_3 S(x_3, x_2)$ ?
- $\alpha(x_1), \alpha(x_5), \alpha(x_6), \alpha(x_7), \dots$  are irrelevant since those variable symbols occur in no formula.
- $\alpha(x_4)$  also is irrelevant: the variable occurs in the formula, but all occurrences are bound by a surrounding quantifier.
- $\rightsquigarrow$  only assignments for free variables  $x_2$  and  $x_3$  relevant

German: gebundene und freie Variablen

# Variables of a Term

## Definition (Variables of a Term)

Let  $t$  be a term. The set of **variables** that occur in  $t$ , written as  $\mathit{var}(t)$ , is defined as follows:

- $\mathit{var}(x) = \{x\}$   
for variable symbols  $x$
- $\mathit{var}(c) = \emptyset$   
for constant symbols  $c$
- $\mathit{var}(f(t_1, \dots, t_k)) = \mathit{var}(t_1) \cup \dots \cup \mathit{var}(t_k)$   
for function terms

**terminology:** A term  $t$  with  $\mathit{var}(t) = \emptyset$  is called **ground term**.

**German:** Grundterm

**example:**  $\mathit{var}(\mathit{product}(x, \mathit{sum}(k, y))) =$

# Free and Bound Variables of a Formula

## Definition (Free Variables)

Let  $\varphi$  be a predicate logic formula. The set of **free variables** of  $\varphi$ , written as  $free(\varphi)$ , is defined as follows:

- $free(P(t_1, \dots, t_k)) = var(t_1) \cup \dots \cup var(t_k)$
- $free((t_1 = t_2)) = var(t_1) \cup var(t_2)$
- $free(\neg\varphi) = free(\varphi)$
- $free((\varphi \wedge \psi)) = free((\varphi \vee \psi)) = free(\varphi) \cup free(\psi)$
- $free(\forall x \varphi) = free(\exists x \varphi) = free(\varphi) \setminus \{x\}$

**Example:**  $free((\forall x_4(R(x_4, x_2) \vee (f(x_3) = x_4)) \vee \exists x_3 S(x_3, x_2)))$   
=

## Closed Formulas/Sentences

**Note:** Let  $\varphi$  be a formula and let  $\alpha$  and  $\beta$  variable assignments with  $\alpha(x) = \beta(x)$  for all free variables  $x$  of  $\varphi$ .

Then  $\mathcal{I}, \alpha \models \varphi$  iff  $\mathcal{I}, \beta \models \varphi$ .

In particular,  $\alpha$  is **completely irrelevant** if  $\text{free}(\varphi) = \emptyset$ .

### Definition (Closed Formulas/Sentences)

A formula  $\varphi$  without free variables (i. e.,  $\text{free}(\varphi) = \emptyset$ ) is called **closed formula** or **sentence**.

If  $\varphi$  is a sentence, then we often write  $\mathcal{I} \models \varphi$  instead of  $\mathcal{I}, \alpha \models \varphi$ , since the definition of  $\alpha$  does not influence whether  $\varphi$  is true under  $\mathcal{I}$  and  $\alpha$  or not.

Formulas with at least one free variable are called **open**.

Closed formulas with no quantifiers are called **ground formulas**.

**German:** geschlossene Formel/Satz, offene Formel,  
Grundformel/variablenfreie Formel

## Closed Formulas/Sentences: Examples

**Question:** Which of the following formulas are sentences?

- $(\text{Block}(b) \vee \neg \text{Block}(b))$
- $(\text{Block}(x) \rightarrow (\text{Block}(x) \vee \neg \text{Block}(y)))$
- $(\text{Block}(a) \wedge \text{Block}(b))$
- $\forall x(\text{Block}(x) \rightarrow \text{Red}(x))$

# Reasoning in Predicate Logic



# Terminology for Formulas

The terminology we introduced for propositional logic equally applies to predicate logic:

- Interpretation  $\mathcal{I}$  and variable assignment  $\alpha$  form a **model** of the formula  $\varphi$  if  $\mathcal{I}, \alpha \models \varphi$ .
- Formula  $\varphi$  is **satisfiable** if  $\mathcal{I}, \alpha \models \varphi$  for at least one  $\mathcal{I}, \alpha$ .
- Formula  $\varphi$  is **falsifiable** if  $\mathcal{I}, \alpha \not\models \varphi$  for at least one  $\mathcal{I}, \alpha$ .
- Formula  $\varphi$  is **valid** if  $\mathcal{I}, \alpha \models \varphi$  for all  $\mathcal{I}, \alpha$ .
- Formula  $\varphi$  is **unsatisfiable** if  $\mathcal{I}, \alpha \not\models \varphi$  for all  $\mathcal{I}, \alpha$ .

**German:** Modell, erfüllbar, falsifizierbar, gültig, unerfüllbar

All concepts can be used for the special case of **sentences**.

In this case we usually omit  $\alpha$ . **Examples:**

- Interpretation  $\mathcal{I}$  is a **model** of a sentence  $\varphi$  if  $\mathcal{I} \models \varphi$ .
- Sentence  $\varphi$  is **unsatisfiable** if  $\mathcal{I} \not\models \varphi$  for all  $\mathcal{I}$ .

# Sets of Formulas: Semantics

## Definition (Satisfied/True Sets of Formulas)

Let  $\mathcal{S}$  be a signature,  $\Phi$  a set of formulas over  $\mathcal{S}$ ,  $\mathcal{I}$  an interpretation for  $\mathcal{S}$  and  $\alpha$  a variable assignment for  $\mathcal{S}$  and the universe of  $\mathcal{I}$ .

We say that  $\mathcal{I}$  and  $\alpha$  **satisfy** the formulas  $\Phi$  (also:  $\Phi$  is **true** under  $\mathcal{I}$  and  $\alpha$ ), written as:  $\mathcal{I}, \alpha \models \Phi$ , if  $\mathcal{I}, \alpha \models \varphi$  for all  $\varphi \in \Phi$ .

**German:**  $\mathcal{I}$  und  $\alpha$  erfüllen  $\Phi$ ,  $\Phi$  ist wahr unter  $\mathcal{I}$  und  $\alpha$

We may again write  $\mathcal{I} \models \Phi$  if all formulas in  $\Phi$  are sentences.

# Logical Equivalence and Logical Consequences

We again use the same concepts and notations as in propositional logic.

- A set of formulas  $\Phi$  logically entails/implies formula  $\psi$ , written as  $\Phi \models \psi$ , if all models of  $\Phi$  are models of  $\psi$ .
- For a single formula  $\varphi$ , we may write  $\varphi \models \psi$  for  $\{\varphi\} \models \psi$ .
- Formulas  $\varphi$  and  $\psi$  are **logically equivalent**, written as  $\varphi \equiv \psi$ , if they have the same models.
  - Note that  $\varphi \equiv \psi$  iff  $\varphi \models \psi$  and  $\psi \models \varphi$ .

# Important Theorems about Logical Consequences

## Theorem (Deduction Theorem)

$KB \cup \{\varphi\} \models \psi$  iff  $KB \models (\varphi \rightarrow \psi)$

German: Deduktionsatz

## Theorem (Contraposition Theorem)

$KB \cup \{\varphi\} \models \neg\psi$  iff  $KB \cup \{\psi\} \models \neg\varphi$

German: Kontrapositionssatz

## Theorem (Contradiction Theorem)

$KB \cup \{\varphi\}$  is unsatisfiable iff  $KB \models \neg\varphi$

German: Widerlegungssatz

These can be proved exactly the same way as in propositional logic.

# Logical Equivalences

- All **logical equivalences of propositional logic** also hold in predicate logic (e. g.,  $(\varphi \vee \psi) \equiv (\psi \vee \varphi)$ ). (**Why?**)
- Additionally the following equivalences and implications hold:

$$(\forall x\varphi \wedge \forall x\psi) \equiv \forall x(\varphi \wedge \psi)$$

$$(\forall x\varphi \vee \forall x\psi) \models \forall x(\varphi \vee \psi)$$

$$(\forall x\varphi \wedge \psi) \equiv \forall x(\varphi \wedge \psi)$$

$$(\forall x\varphi \vee \psi) \equiv \forall x(\varphi \vee \psi)$$

$$\neg\forall x\varphi \equiv \exists x\neg\varphi$$

$$\exists x(\varphi \vee \psi) \equiv (\exists x\varphi \vee \exists x\psi)$$

$$\exists x(\varphi \wedge \psi) \models (\exists x\varphi \wedge \exists x\psi)$$

$$(\exists x\varphi \vee \psi) \equiv \exists x(\varphi \vee \psi)$$

$$(\exists x\varphi \wedge \psi) \equiv \exists x(\varphi \wedge \psi)$$

$$\neg\exists x\varphi \equiv \forall x\neg\varphi$$

but not the converse

if  $x \notin \text{free}(\psi)$

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# Normal Forms (1)

Analogously to DNF and CNF for propositional logic there are several normal forms for predicate logic, such as

- **negation normal form (NNF):**  
negation symbols ( $\neg$ ) are only allowed in front of atoms
- **prenex normal form:**  
quantifiers must form the outermost part of the formula
- **Skolem normal form:**  
prenex normal form without existential quantifiers

**German:** Negationsnormalform, Pränexnormalform, Skolemnormalform

## Normal Forms (2)

Efficient methods transform formula  $\varphi$

- into an **equivalent** formula in **negation normal form**,
- into an **equivalent** formula in **prenex normal form**, or
- into an **equisatisfiable** formula in **Skolem normal form**.

**German:** erfüllbarkeitsäquivalent

# Inference Rules and Calculi

There exist correct and complete **proof systems** (**calculi**) for predicate logic.

- An example is the **natural deduction** calculus.
- This is (essentially) Gödel's Completeness Theorem (1929).
- However, one can show that correct and complete algorithms that prove that a given formula **does not** follow from a given set of formulas **cannot exist**.
- How are these statements reconcilable?



# Summary and Outlook

# Summary

- **Predicate logic** is more expressive than propositional logic and allows statements over **objects** and their **properties**.
- Objects are described by **terms** that are built from variable, constant and function symbols.
- Properties and relations are described by **formulas** that are built from predicates, quantifiers and the usual logical operators.
- **Bound** vs. **free** variables: to decide if  $\mathcal{I}, \alpha \models \varphi$ , only free variables in  $\alpha$  matter
- **Sentences** (closed formulas): formulas without free variables

# Summary

Once the basic definitions are in place, predicate logic can be developed in the same way as propositional logic:

- logical consequence
- deduction theorem etc.
- logical equivalences
- normal forms
- inference rules, proof systems, resolution

## Other Logics (1)

- We considered **first-order** predicate logic.
- **Second-order** predicate logic allows quantifying over predicate symbols.
- There are intermediate steps, e. g., monadic second-order logic (all quantified predicates are unary) and **description logics** (foundation of the semantic web).

# Second-Order Logic Example

## Second-order logic example:

- “ $T$  is the transitive closure of  $R$ ”
- conjunction of
  - $\forall x \forall y (R(x, y) \rightarrow T(x, y))$   
“ $T$  is a superset of  $R$ ”
  - $\forall x \forall y \forall z ((T(x, y) \wedge T(y, z)) \rightarrow T(x, z))$   
“ $T$  is transitive”
  - $\forall Q ((\forall x \forall y (R(x, y) \rightarrow Q(x, y)) \wedge$   
 $\forall x \forall y \forall z ((Q(x, y) \wedge Q(y, z)) \rightarrow Q(x, z)))$   
 $\rightarrow \forall x \forall y (T(x, y) \rightarrow Q(x, y)))$   
“All supersets  $Q$  of  $R$  that are transitive are supersets of  $T$ ”
- impossible to express in first-order logic

## Other Logics (2)

- **Modal logics** have new operators  $\Box$  and  $\Diamond$ .
  - classical meaning:  $\Box\varphi$  for “ $\varphi$  is necessary”,  
 $\Diamond\varphi$  for “ $\varphi$  is possible”.
  - temporal logic:  $\Box\varphi$  for “ $\varphi$  is always true in the future”,  
 $\Diamond\varphi$  for “ $\varphi$  is true at some point in the future”
  - epistemic logic:  $\Box\varphi$  for “ $\varphi$  is known”,  
 $\Diamond\varphi$  for “ $\varphi$  is possible”
  - doxastic logic:  $\Box\varphi$  for “ $\varphi$  is believed”,  
 $\Diamond\varphi$  for “ $\varphi$  is considered possible”
  - deontic logic:  $\Box\varphi$  for “ $\varphi$  is obligatory”,  
 $\Diamond\varphi$  for “ $\varphi$  is permitted”
  - ...
- very important in computer-aided verification

## Other Logics (3)

- In **fuzzy logic**, formulas are not true or false but have values between 0 and 1.
- **Intuitionist logic** is “constructive” and excludes indirect proof methods such as the principle of the excluded third.
- **Non-monotonic logics** have rules with exceptions (e.g., default logic, cumulative logic).
- ... and there is a lot more