Discrete Mathematics in Computer Science D6. Advanced Concepts in Predicate Logic and Outlook

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December 18, 2023

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Free and Bound Variables

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# D6.1 Free and Bound Variables

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## Free and Bound Variables: Motivation

#### Question:

- Consider a signature with variable symbols {x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>,...}
  and an interpretation *I*.
- ▶ Which parts of the definition of  $\alpha$  are relevant to decide whether  $\mathcal{I}, \alpha \models (\forall x_4(\mathsf{R}(x_4, x_2) \lor (\mathsf{f}(x_3) = x_4)) \lor \exists x_3\mathsf{S}(x_3, x_2))$ ?
- α(x<sub>1</sub>), α(x<sub>5</sub>), α(x<sub>6</sub>), α(x<sub>7</sub>), ... are irrelevant since those variable symbols occur in no formula.
- α(x<sub>4</sub>) also is irrelevant: the variable occurs in the formula, but all occurrences are bound by a surrounding quantifier.
- $\blacktriangleright$   $\rightarrow$  only assignments for free variables  $x_2$  and  $x_3$  relevant

German: gebundene und freie Variablen

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#### Free and Bound Variables

#### Variables of a Term

Definition (Variables of a Term)

Let t be a term. The set of variables that occur in t, written as var(t), is defined as follows:

- var(x) = {x}
  for variable symbols x
- Var(c) = ∅ for constant symbols c
- ►  $var(f(t_1,...,t_k)) = var(t_1) \cup \cdots \cup var(t_k)$ for function terms

terminology: A term t with  $var(t) = \emptyset$  is called ground term. German: Grundterm

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example: var(product(x, sum(k, y))) =
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### Closed Formulas/Sentences

Note: Let  $\varphi$  be a formula and let  $\alpha$  and  $\beta$  variable assignments with  $\alpha(x) = \beta(x)$  for all free variables x of  $\varphi$ . Then  $\mathcal{I}, \alpha \models \varphi$  iff  $\mathcal{I}, \beta \models \varphi$ .

In particular,  $\alpha$  is completely irrelevant if  $free(\varphi) = \emptyset$ .

#### Definition (Closed Formulas/Sentences)

A formula  $\varphi$  without free variables (i. e., *free*( $\varphi$ ) =  $\emptyset$ ) is called closed formula or sentence.

If  $\varphi$  is a sentence, then we often write  $\mathcal{I} \models \varphi$ instead of  $\mathcal{I}, \alpha \models \varphi$ , since the definition of  $\alpha$  does not influence whether  $\varphi$  is true under  $\mathcal{I}$  and  $\alpha$  or not.

Formulas with at least one free variable are called open.

Closed formulas with no quantifiers are called ground formulas.

German: geschlossene Formel/Satz, offene Formel, Grundformel/variablenfreie Formel A. Helmert, G. Röger (University of Basel) Discrete Mathematics in Computer Science December 18, 2023 D6. Advanced Concepts in Predicate Logic and Outlook

### Free and Bound Variables of a Formula

Definition (Free Variables)

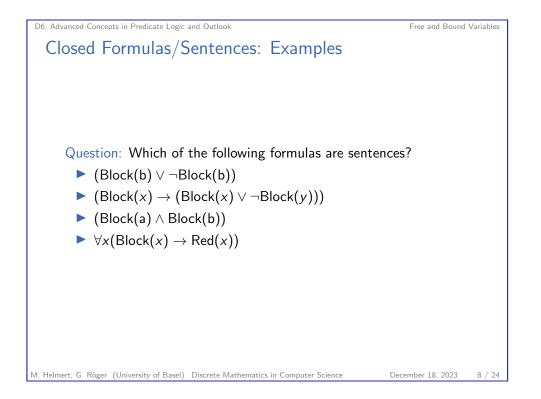
Let  $\varphi$  be a predicate logic formula. The set of free variables of  $\varphi$ , written as *free*( $\varphi$ ), is defined as follows:

- $free(P(t_1,\ldots,t_k)) = var(t_1) \cup \cdots \cup var(t_k)$
- $\blacktriangleright free((t_1 = t_2)) = var(t_1) \cup var(t_2)$
- free( $\neg \varphi$ ) = free( $\varphi$ )
- $\blacktriangleright \ \textit{free}((\varphi \land \psi)) = \textit{free}((\varphi \lor \psi)) = \textit{free}(\varphi) \cup \textit{free}(\psi)$
- $\models free(\forall x \varphi) = free(\exists x \varphi) = free(\varphi) \setminus \{x\}$

Example: *free*( $(\forall x_4(R(x_4, x_2) \lor (f(x_3) = x_4)) \lor \exists x_3S(x_3, x_2)))$ =

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# D6.2 Reasoning in Predicate Logic

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Sets of Formulas: Semantics

Definition (Satisfied/True Sets of Formulas)

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Let S be a signature,  $\Phi$  a set of formulas over S,  $\mathcal{I}$  an interpretation for S and  $\alpha$  a variable assignment for Sand the universe of  $\mathcal{I}$ .

We say that  $\mathcal{I}$  and  $\alpha$  satisfy the formulas  $\Phi$ (also:  $\Phi$  is true under  $\mathcal{I}$  and  $\alpha$ ), written as:  $\mathcal{I}, \alpha \models \Phi$ , if  $\mathcal{I}, \alpha \models \varphi$  for all  $\varphi \in \Phi$ .

German:  $\mathcal{I}$  und  $\alpha$  erfüllen  $\Phi$ ,  $\Phi$  ist wahr unter  $\mathcal{I}$  und  $\alpha$ 

We may again write  $\mathcal{I} \models \Phi$  if all formulas in  $\Phi$  are sentences.

#### Terminology for Formulas

The terminology we introduced for propositional logic equally applies to predicate logic:

- Interpretation *I* and variable assignment α form a model of the formula φ if *I*, α ⊨ φ.
- Formula  $\varphi$  is satisfiable if  $\mathcal{I}, \alpha \models \varphi$  for at least one  $\mathcal{I}, \alpha$ .
- Formula  $\varphi$  is falsifiable if  $\mathcal{I}, \alpha \not\models \varphi$ . for at least one  $\mathcal{I}, \alpha$
- Formula  $\varphi$  is valid if  $\mathcal{I}, \alpha \models \varphi$  for all  $\mathcal{I}, \alpha$ .
- Formula  $\varphi$  is unsatisfiable if  $\mathcal{I}, \alpha \not\models \varphi$  for all  $\mathcal{I}, \alpha$ .

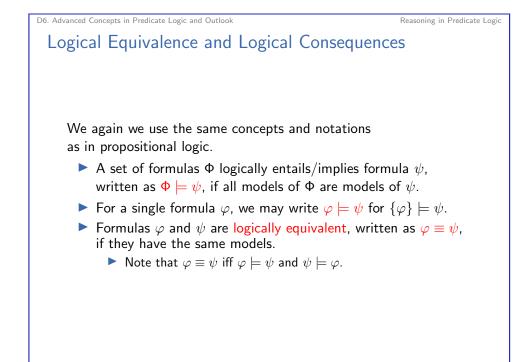
German: Modell, erfüllbar, falsifizierbar, gültig, unerfüllbar

All concepts can be used for the special case of sentences. In this case we usually omit  $\alpha$ . Examples:

- Interpretation  $\mathcal{I}$  is a model of a sentence  $\varphi$  if  $\mathcal{I} \models \varphi$ .
- Sentence  $\varphi$  is unsatisfiable if  $\mathcal{I} \not\models \varphi$  for all  $\mathcal{I}$ .

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Reasoning in Predicate Logic

## Important Theorems about Logical Consequences

Theorem (Deduction Theorem) KB  $\cup$  { $\varphi$ }  $\models \psi$  *iff* KB  $\models$  ( $\varphi \rightarrow \psi$ )

German: Deduktionssatz

Theorem (Contraposition Theorem) KB  $\cup \{\varphi\} \models \neg \psi \text{ iff } KB \cup \{\psi\} \models \neg \varphi$ 

German: Kontrapositionssatz

Theorem (Contradiction Theorem)

 $\mathsf{KB} \cup \{\varphi\} \text{ is unsatisfiable iff } \mathsf{KB} \models \neg \varphi$ 

German: Widerlegungssatz

These can be proved exactly the same way as in propositional logic.

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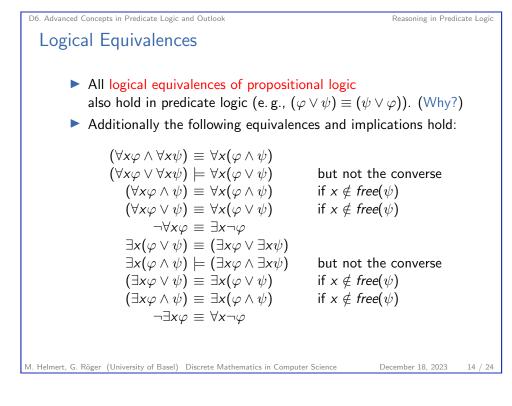
Normal Forms (1)

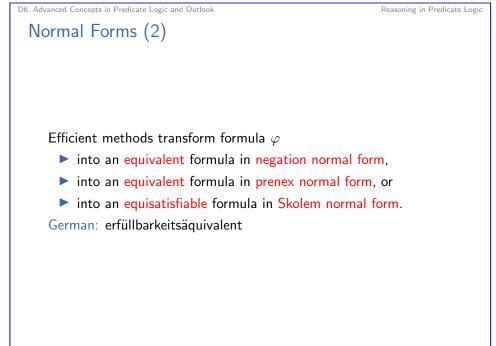
Analogously to DNF and CNF for propositional logic there are several normal forms for predicate logic, such as

- negation normal form (NNF):
  negation symbols (¬) are only allowed in front of atoms
- prenex normal form: quantifiers must form the outermost part of the formula
- Skolem normal form:

prenex normal form without existential quantifiers German: Negationsnormalform, Pränexnormalform,

Skolemnormalform





#### D6. Advanced Concepts in Predicate Logic and Outlook

Reasoning in Predicate Logic

Summary and Outlook

#### Inference Rules and Calculi

There exist correct and complete proof systems (calculi) for predicate logic.

- An example is the natural deduction calculus.
- ▶ This is (essentially) Gödel's Completeness Theorem (1929).
- However, one can show that correct and complete algorithms that prove that a given formula does not follow from a given set of formulas cannot exist.
- How are these statements reconcilable?

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Summary

- Predicate logic is more expressive than propositional logic and allows statements over objects and their properties.
- Objects are described by terms that are built from variable, constant and function symbols.
- Properties and relations are described by formulas that are built from predicates, quantifiers and the usual logical operators.
- Bound vs. free variables: to decide if *I*, α ⊨ φ, only free variables in α matter
- Sentences (closed formulas): formulas without free variables



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Summary

Once the basic definitions are in place, predicate logic can be developed in the same way as propositional logic:

- logical consequence
- deduction theorem etc.
- logical equivalences
- normal forms
- ▶ inference rules, proof systems, resolution

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