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D5. Syntax and Semantics of Predicate Logic

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# D5.1 Syntax of Predicate Logic

Discrete Mathematics in Computer Science December 11/13, 2023 — D5. Syntax and Semantics of Predicate Logic

D5.1 Syntax of Predicate Logic

D5.2 Semantics of Predicate Logic

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### D5. Syntax and Semantics of Predicate Logic Limits of Propositional Logic

Cannot be expressed well in propositional logic:

- "Everyone who does the exercises passes the exam."
- "If someone with administrator privileges presses 'delete', all data is gone."

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- "Everyone has a mother."
- "If someone is the father of some person, the person is his child."

▷ need more expressive logic

→ predicate logic (a.k.a. first-order logic)

German: Prädikatenlogik (erster Stufe)

#### D5. Syntax and Semantics of Predicate Logic

### Syntax: Building Blocks

Signatures define allowed symbols. analogy: atom set A in propositional logic

- Terms are associated with objects by the semantics. no analogy in propositional logic
- Formulas are associated with truth values (true or false) by the semantics.
   analogy: formulas in propositional logic

German: Signatur, Term, Formel

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Signatures: Terminology and Conventions

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#### terminology:

- k-ary (function or predicate) symbol: symbol s with arity ar(s) = k.
- also: unary, binary, ternary

German: k-stellig, unär, binär, ternär

conventions (in this course):

- variable symbols written in *italics*, other symbols upright.
- predicate symbols begin with capital letter, other symbols with lower-case letters

### Signatures: Definition

#### Definition (Signature)

A signature (of predicate logic) is a 4-tuple  $S = \langle V, C, F, P \rangle$  consisting of the following four disjoint sets:

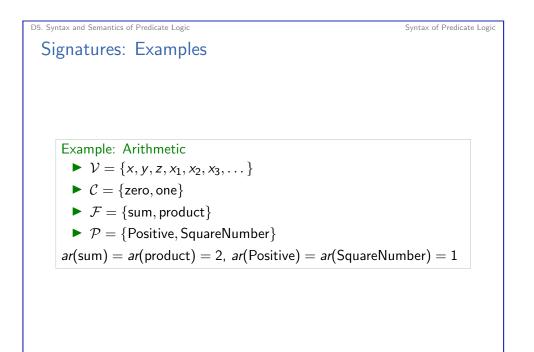
- $\blacktriangleright$  a finite or countable set  ${\cal V}$  of variable symbols
- $\blacktriangleright$  a finite or countable set  ${\cal C}$  of constant symbols
- $\blacktriangleright$  a finite or countable set  $\mathcal{F}$  of function symbols
- a finite or countable set *P* of predicate symbols (or relation symbols)

Every function symbol  $f \in \mathcal{F}$  and predicate symbol  $P \in \mathcal{P}$  has an associated arity  $ar(f), ar(P) \in \mathbb{N}_1$  (number of arguments).

German: Variablen-, Konstanten-, Funktions-, Prädikat- und Relationssymbole; Stelligkeit

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### Signatures: Examples

#### Example: Genealogy

- $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- $C = \{$ roger-federer, lisa-simpson $\}$
- $\blacktriangleright \mathcal{F} = \emptyset$
- $\blacktriangleright \mathcal{P} = \{\mathsf{Female}, \mathsf{Male}, \mathsf{Parent}\}$

$$ar(Female) = ar(Male) = 1, ar(Parent) = 2$$

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### Formulas: Definition

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#### Definition (Formula)

For a signature  $S = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  the set of predicate logic formulas (over S) is inductively defined as follows:

- If t<sub>1</sub>,..., t<sub>k</sub> are terms (over S) and P ∈ P is a k-ary predicate symbol, then the atomic formula (or the atom) P(t<sub>1</sub>,..., t<sub>k</sub>) is a formula over S.
- If  $t_1$  and  $t_2$  are terms (over S), then the identity  $(t_1 = t_2)$  is a formula over S.
- If x ∈ V is a variable symbol and φ a formula over S, then the universal quantification ∀x φ and the existential quantification ∃x φ are formulas over S.

German: atomare Formel, Atom, Identität, Allquantifizierung, Existenzquantifizierung Syntax of Predicate Logic

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### Terms: Definition

#### Definition (Term)

Let  $S = \langle V, C, F, P \rangle$  be a signature. A term (over S) is inductively constructed according to the following rules:

- Every variable symbol  $\mathbf{v} \in \mathcal{V}$  is a term.
- Every constant symbol  $\mathbf{c} \in \mathcal{C}$  is a term.
- ▶ If  $t_1, ..., t_k$  are terms and  $f \in \mathcal{F}$  is a function symbol with arity k, then  $f(t_1, ..., t_k)$  is a term.

#### German: Term

#### examples:

- ► x4
- lisa-simpson
- sum(x<sub>3</sub>, product(one, x<sub>5</sub>))

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#### D5. Syntax and Semantics of Predicate Logic

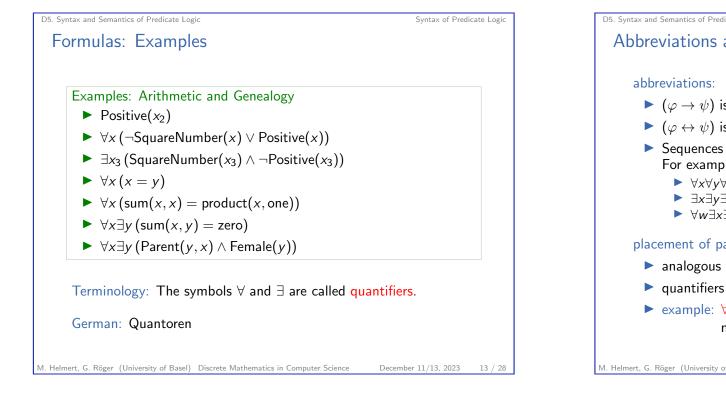
## Formulas: Definition

#### Definition (Formula)

For a signature  $S = \langle V, C, F, P \rangle$  the set of predicate logic formulas (over S) is inductively defined as follows:

- If  $\varphi$  is a formula over S, then so is its negation  $\neg \varphi$ .
- If φ and ψ are formulas over S, then so are the conjunction (φ ∧ ψ) and the disjunction (φ ∨ ψ).

#### German: Negation, Konjunktion, Disjunktion



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### Exercise

 $S = \langle \{x, y, z\}, \{c\}, \{f, g, h\}, \{Q, R, S\} \rangle$  with ar(f) = 3, ar(g) = ar(h) = 1, ar(Q) = 2, ar(R) = ar(S) = 1

- $\blacktriangleright$  f(x, y)
- (g(x) = R(y))
- $\blacktriangleright$  (g(x) = f(y, c, h(x)))
- $\blacktriangleright$  (R(x)  $\land \forall x S(x)$ )
- $\blacktriangleright \forall c Q(c, x)$
- $\blacktriangleright (\forall x \exists y (g(x) = y) \lor (h(x) = c))$

Which expressions are syntactically correct formulas or terms for S? What kind of term/formula?

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### Abbreviations and Placement of Parentheses by Convention

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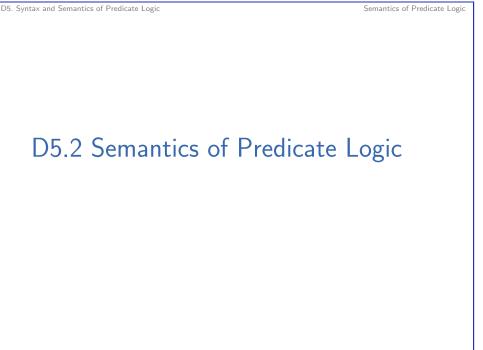
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- $(\varphi \to \psi)$  is an abbreviation for  $(\neg \varphi \lor \psi)$ .
- $(\varphi \leftrightarrow \psi)$  is an abbreviation for  $((\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi))$ .
- Sequences of the same quantifier can be abbreviated. For example:
  - $\blacktriangleright \forall x \forall y \forall z \varphi \rightsquigarrow \forall x y z \varphi$
  - $\blacktriangleright \exists x \exists y \exists z \varphi \rightsquigarrow \exists x y z \varphi$
  - $\blacktriangleright \forall w \exists x \exists v \forall z \varphi \rightsquigarrow \forall w \exists x v \forall z \varphi$

#### placement of parentheses by convention:

- analogous to propositional logic
- $\blacktriangleright$  quantifiers  $\forall$  and  $\exists$  bind more strongly than anything else.
- example:  $\forall x P(x) \rightarrow Q(x)$  corresponds to  $(\forall x P(x) \rightarrow Q(x))$ , not  $\forall x (\mathsf{P}(x) \to \mathsf{Q}(x)).$

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	Interpretations and Variable Assignments
	Let $\mathcal{S}=\langle\mathcal{V},\mathcal{C},\mathcal{F},\mathcal{P} angle$ be a signature.
interpretations in propositional logic:	Definition (Interpretation, Variable Assignment)
	An interpretation (for S) is a pair $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ of:
<ul> <li>truth assignments for the propositional variables</li> <li>There are no propositional variables in predicate logic.</li> <li>instead: interpretation determines meaning</li> </ul>	<ul> <li>a non-empty set <i>U</i> called the universe and</li> <li>a function <sup>I</sup> that assigns a meaning to the constant, function, and predicate symbols:</li> </ul>
<ul> <li>of the constant, function and predicate symbols.</li> <li>meaning of variable symbols not determined by interpretation but by separate variable assignment</li> </ul>	<ul> <li>c<sup>I</sup> ∈ U for constant symbols c ∈ C</li> <li>f<sup>I</sup> : U<sup>k</sup> → U for k-ary function symbols f ∈ F</li> <li>P<sup>I</sup> ⊆ U<sup>k</sup> for k-ary predicate symbols P ∈ P</li> </ul>
	A variable assignment (for $S$ and universe $U$ ) is a function $\alpha : \mathcal{V} \to U$ .
	German: Interpretation, Universum (or Grundmenge), Variablenzuweisung
Syntax and Semantics of Predicate Logic Semantics of Predicate Logic <b>nterpretations and Variable Assignments: Example</b>	D5. Syntax and Semantics of Predicate Logic Semantics of Pre Semantics: Informally
	Semantics: Informally Example: $(\forall x(Block(x) \rightarrow Red(x)) \land Block(a))$ "For all objects x: if x is a block, then x is red. Also, the object called a is a block."
Interpretations and Variable Assignments: Example Example signature: $S = \langle V, C, F, P \rangle$ with $V = \{x, y, z\}$ , $C = \{\text{zero, one}\}, F = \{\text{sum, product}\}, P = \{\text{SquareNumber}\}$	Semantics: Informally Example: $(\forall x(Block(x) \rightarrow Red(x)) \land Block(a))$ "For all objects x: if x is a block, then x is red. Also, the object called a is a block." Terms are interpreted as objects.
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Interpretations and Variable Assignments: Example Example signature: $S = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{V} = \{x, y, z\}$ , $\mathcal{C} = \{\text{zero, one}\}, \mathcal{F} = \{\text{sum, product}\}, \mathcal{P} = \{\text{SquareNumber}\}$ ar(sum) = ar(product) = 2, ar(SquareNumber) = 1 $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with $\mathbf{U} = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$ $\mathbf{V} = \text{zero}^{\mathcal{I}} = u_0$ $\mathbf{V} = \text{one}^{\mathcal{I}} = u_1$ $\mathbf{U} = \text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \mod 7}$ for all $i, j \in \{0, \dots, 6\}$	<ul> <li>Semantics: Informally</li> <li>Example: (∀x(Block(x) → Red(x)) ∧ Block(a)) "For all objects x: if x is a block, then x is red. Also, the object called a is a block."</li> <li>Terms are interpreted as objects.</li> <li>Unary predicates denote properties of objects (to be a block, to be red, to be a square number,).</li> <li>General predicates denote relations between objects (to be someone's child, to have a common divisor,).</li> <li>Universally quantified formulas ("∀") are true if they hold for every object in the universe.</li> </ul>

Semantics of Predicate Logic

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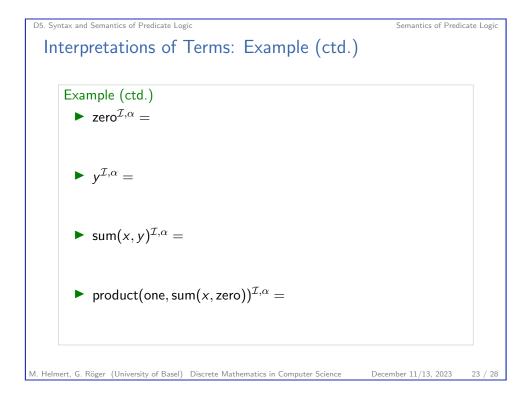
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### Interpretations of Terms

Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature.

Definition (Interpretation of a Term) Let  $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  be an interpretation for  $\mathcal{S}$ , and let  $\alpha$  be a variable assignment for  $\mathcal{S}$  and universe U. Let t be a term over  $\mathcal{S}$ . The interpretation of t under  $\mathcal{I}$  and  $\alpha$ , written as  $t^{\mathcal{I},\alpha}$ , is the element of the universe U defined as follows: If t = x with  $x \in \mathcal{V}$  (t is a variable term):  $x^{\mathcal{I},\alpha} = \alpha(x)$ If t = c with  $c \in \mathcal{C}$  (t is a constant term):  $c^{\mathcal{I},\alpha} = c^{\mathcal{I}}$ If  $t = f(t_1, \dots, t_k)$  (t is a function term):  $f(t_1, \dots, t_k)^{\mathcal{I},\alpha} = f^{\mathcal{I}}(t_1^{\mathcal{I},\alpha}, \dots, t_k^{\mathcal{I},\alpha})$ 

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Semantics of Predicate Logic

### Interpretations of Terms: Example

#### Example

signature:  $S = \langle V, C, F, P \rangle$ with  $V = \{x, y, z\}$ ,  $C = \{\text{zero, one}\}$ ,  $F = \{\text{sum, product}\}$ , ar(sum) = ar(product) = 2

 $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle \text{ with}$   $\mathbf{\vdash} U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$   $\mathbf{\vdash} \text{ zero}^{\mathcal{I}} = u_0$   $\mathbf{\vdash} \text{ one}^{\mathcal{I}} = u_1$   $\mathbf{\vdash} \text{ sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \mod 7} \text{ for all } i, j \in \{0, \dots, 6\}$   $\mathbf{\vdash} \text{ product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \mod 7} \text{ for all } i, j \in \{0, \dots, 6\}$   $\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$ 

D5. Syntax and Semantics of Predicate Logic Semantics of Predicate Logic Semantics of Predicate Logic Formulas Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature. Definition (Formula is Satisfied or True) Let  $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  be an interpretation for  $\mathcal{S}$ , and let  $\alpha$  be a variable assignment for S and universe U. We say that  $\mathcal{I}$  and  $\alpha$  satisfy a predicate logic formula  $\varphi$ (also:  $\varphi$  is true under  $\mathcal{I}$  and  $\alpha$ ), written:  $\mathcal{I}, \alpha \models \varphi$ , according to the following inductive rules:  $\mathcal{I}, \alpha \models \mathsf{P}(t_1, \ldots, t_k) \quad \text{iff } \langle t_1^{\mathcal{I}, \alpha}, \ldots, t_k^{\mathcal{I}, \alpha} \rangle \in \mathsf{P}^{\mathcal{I}}$  $\mathcal{I}, \alpha \models (t_1 = t_2) \quad \text{iff} \ t_1^{\mathcal{I}, \alpha} = t_2^{\mathcal{I}, \alpha}$  $\mathcal{I}, \alpha \models \neg \varphi \quad \text{iff } \mathcal{I}, \alpha \not\models \varphi$  $\mathcal{I}, \alpha \models (\varphi \land \psi)$  iff  $\mathcal{I}, \alpha \models \varphi$  and  $\mathcal{I}, \alpha \models \psi$  $\mathcal{I}, \alpha \models (\varphi \lor \psi) \quad \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi$ German:  $\mathcal{I}$  und  $\alpha$  erfüllen  $\varphi$  (also:  $\varphi$  ist wahr unter  $\mathcal{I}$  und  $\alpha$ )

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### Semantics of Predicate Logic Formulas

Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature.

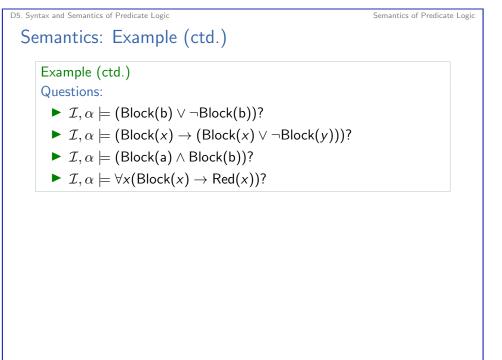
Definition (Formula is Satisfied or True)

 $\begin{array}{ll} \mathcal{I}, \alpha \models \forall x \varphi & \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for all } u \in U \\ \mathcal{I}, \alpha \models \exists x \varphi & \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for at least one } u \in U \end{array}$ 

where  $\alpha[x := u]$  is the same variable assignment as  $\alpha$ , except that it maps variable x to the value u. Formally:

 $(\alpha[x := u])(z) = \begin{cases} u & \text{if } z = x \\ \alpha(z) & \text{if } z \neq x \end{cases}$ 

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### Semantics: Example

#### Example

signature:  $S = \langle V, C, F, P \rangle$ with  $V = \{x, y, z\}$ ,  $C = \{a, b\}$ ,  $F = \emptyset$ ,  $P = \{Block, Red\}$ , ar(Block) = ar(Red) = 1.

