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D5. Syntax and Semantics of Predicate Logic

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Syntax of Predicate Logic

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D5.1 Syntax of Predicate Logic

Discrete Mathematics in Computer Science December 11/13, 2023 — D5. Syntax and Semantics of Predicate Logic

D5.1 Syntax of Predicate Logic

D5.2 Semantics of Predicate Logic

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D5. Syntax and Semantics of Predicate Logic Limits of Propositional Logic

Cannot be expressed well in propositional logic:

- "Everyone who does the exercises passes the exam."
- "If someone with administrator privileges presses 'delete', all data is gone."

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- "Everyone has a mother."
- "If someone is the father of some person, the person is his child."

▷ need more expressive logic

→ predicate logic (a.k.a. first-order logic)

German: Prädikatenlogik (erster Stufe)

D5. Syntax and Semantics of Predicate Logic

Syntax: Building Blocks

Signatures define allowed symbols. analogy: atom set A in propositional logic

- Terms are associated with objects by the semantics. no analogy in propositional logic
- Formulas are associated with truth values (true or false) by the semantics.
 analogy: formulas in propositional logic

German: Signatur, Term, Formel

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Signatures: Terminology and Conventions

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terminology:

- k-ary (function or predicate) symbol: symbol s with arity ar(s) = k.
- also: unary, binary, ternary

German: k-stellig, unär, binär, ternär

conventions (in this course):

- variable symbols written in *italics*, other symbols upright.
- predicate symbols begin with capital letter, other symbols with lower-case letters

Signatures: Definition

Definition (Signature)

A signature (of predicate logic) is a 4-tuple $S = \langle V, C, F, P \rangle$ consisting of the following four disjoint sets:

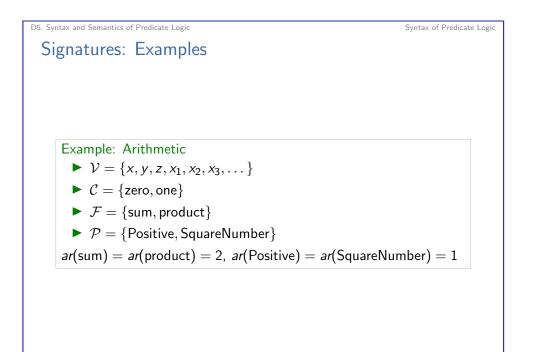
- \blacktriangleright a finite or countable set ${\cal V}$ of variable symbols
- \blacktriangleright a finite or countable set ${\cal C}$ of constant symbols
- \blacktriangleright a finite or countable set \mathcal{F} of function symbols
- a finite or countable set *P* of predicate symbols (or relation symbols)

Every function symbol $f \in \mathcal{F}$ and predicate symbol $P \in \mathcal{P}$ has an associated arity $ar(f), ar(P) \in \mathbb{N}_1$ (number of arguments).

German: Variablen-, Konstanten-, Funktions-, Prädikat- und Relationssymbole; Stelligkeit

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Signatures: Examples

Example: Genealogy

- $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- $C = \{$ roger-federer, lisa-simpson $\}$
- $\blacktriangleright \mathcal{F} = \emptyset$
- $\blacktriangleright \mathcal{P} = \{\mathsf{Female}, \mathsf{Male}, \mathsf{Parent}\}$

$$ar(Female) = ar(Male) = 1, ar(Parent) = 2$$

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Formulas: Definition

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Definition (Formula)

For a signature $S = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas (over S) is inductively defined as follows:

- If t₁,..., t_k are terms (over S) and P ∈ P is a k-ary predicate symbol, then the atomic formula (or the atom) P(t₁,..., t_k) is a formula over S.
- If t_1 and t_2 are terms (over S), then the identity $(t_1 = t_2)$ is a formula over S.
- If x ∈ V is a variable symbol and φ a formula over S, then the universal quantification ∀x φ and the existential quantification ∃x φ are formulas over S.

German: atomare Formel, Atom, Identität, Allquantifizierung, Existenzquantifizierung Syntax of Predicate Logic

Syntax of Predicate Logic

Terms: Definition

Definition (Term)

Let $S = \langle V, C, F, P \rangle$ be a signature. A term (over S) is inductively constructed according to the following rules:

- Every variable symbol $\mathbf{v} \in \mathcal{V}$ is a term.
- Every constant symbol $\mathbf{c} \in \mathcal{C}$ is a term.
- ▶ If $t_1, ..., t_k$ are terms and $f \in \mathcal{F}$ is a function symbol with arity k, then $f(t_1, ..., t_k)$ is a term.

German: Term

examples:

- ► x4
- lisa-simpson
- sum(x₃, product(one, x₅))

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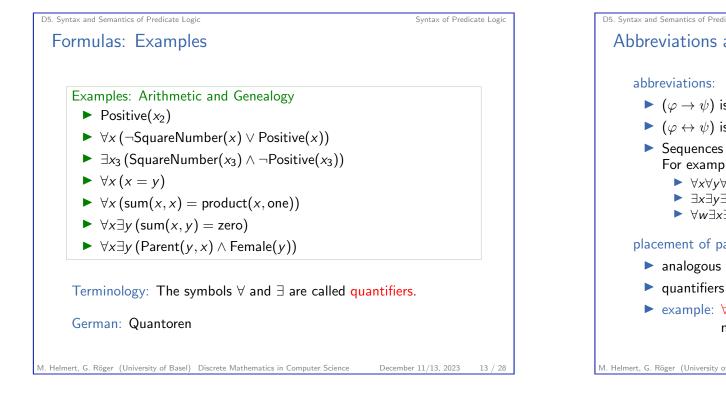
Formulas: Definition

Definition (Formula)

For a signature $S = \langle V, C, F, P \rangle$ the set of predicate logic formulas (over S) is inductively defined as follows:

- If φ is a formula over S, then so is its negation $\neg \varphi$.
- If φ and ψ are formulas over S, then so are the conjunction (φ ∧ ψ) and the disjunction (φ ∨ ψ).

German: Negation, Konjunktion, Disjunktion



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Exercise

 $S = \langle \{x, y, z\}, \{c\}, \{f, g, h\}, \{Q, R, S\} \rangle$ with ar(f) = 3, ar(g) = ar(h) = 1, ar(Q) = 2, ar(R) = ar(S) = 1

- \blacktriangleright f(x, y)
- (g(x) = R(y))
- \blacktriangleright (g(x) = f(y, c, h(x)))
- \blacktriangleright (R(x) $\land \forall x S(x)$)
- $\blacktriangleright \forall c Q(c, x)$
- $\blacktriangleright (\forall x \exists y (g(x) = y) \lor (h(x) = c))$

Which expressions are syntactically correct formulas or terms for S? What kind of term/formula?

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Abbreviations and Placement of Parentheses by Convention

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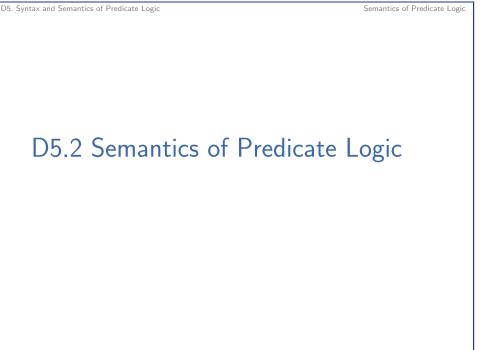
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- $(\varphi \to \psi)$ is an abbreviation for $(\neg \varphi \lor \psi)$.
- $(\varphi \leftrightarrow \psi)$ is an abbreviation for $((\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi))$.
- Sequences of the same quantifier can be abbreviated. For example:
 - $\blacktriangleright \forall x \forall y \forall z \varphi \rightsquigarrow \forall x y z \varphi$
 - $\blacktriangleright \exists x \exists y \exists z \varphi \rightsquigarrow \exists x y z \varphi$
 - $\blacktriangleright \forall w \exists x \exists v \forall z \varphi \rightsquigarrow \forall w \exists x v \forall z \varphi$

placement of parentheses by convention:

- analogous to propositional logic
- \blacktriangleright quantifiers \forall and \exists bind more strongly than anything else.
- example: $\forall x P(x) \rightarrow Q(x)$ corresponds to $(\forall x P(x) \rightarrow Q(x))$, not $\forall x (\mathsf{P}(x) \to \mathsf{Q}(x)).$

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	Interpretations and Variable Assignments
	Let $\mathcal{S}=\langle\mathcal{V},\mathcal{C},\mathcal{F},\mathcal{P} angle$ be a signature.
interpretations in propositional logic:	Definition (Interpretation, Variable Assignment)
	An interpretation (for S) is a pair $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ of:
 truth assignments for the propositional variables There are no propositional variables in predicate logic. instead: interpretation determines meaning 	 a non-empty set <i>U</i> called the universe and a function ^I that assigns a meaning to the constant, function, and predicate symbols:
 of the constant, function and predicate symbols. meaning of variable symbols not determined by interpretation but by separate variable assignment 	 c^I ∈ U for constant symbols c ∈ C f^I : U^k → U for k-ary function symbols f ∈ F P^I ⊆ U^k for k-ary predicate symbols P ∈ P
	A variable assignment (for S and universe U) is a function $\alpha : \mathcal{V} \to U$.
	German: Interpretation, Universum (or Grundmenge), Variablenzuweisung
Syntax and Semantics of Predicate Logic Semantics of Predicate Logic nterpretations and Variable Assignments: Example	D5. Syntax and Semantics of Predicate Logic Semantics of Pre Semantics: Informally
	Semantics: Informally Example: $(\forall x(Block(x) \rightarrow Red(x)) \land Block(a))$ "For all objects x: if x is a block, then x is red. Also, the object called a is a block."
Interpretations and Variable Assignments: Example Example signature: $S = \langle V, C, F, P \rangle$ with $V = \{x, y, z\}$, $C = \{\text{zero, one}\}, F = \{\text{sum, product}\}, P = \{\text{SquareNumber}\}$	Semantics: Informally Example: $(\forall x(Block(x) \rightarrow Red(x)) \land Block(a))$ "For all objects x: if x is a block, then x is red. Also, the object called a is a block." Terms are interpreted as objects.
Interpretations and Variable Assignments: Example Example signature: $S = \langle V, C, F, P \rangle$ with $V = \{x, y, z\}$, $C = \{\text{zero, one}\}, F = \{\text{sum, product}\}, P = \{\text{SquareNumber}\}$ ar(sum) = ar(product) = 2, ar(SquareNumber) = 1	Semantics: Informally Example: $(\forall x (Block(x) \rightarrow Red(x)) \land Block(a))$ "For all objects x: if x is a block, then x is red. Also, the object called a is a block."
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Interpretations and Variable Assignments: Example Example signature: $S = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\text{zero, one}\}, \mathcal{F} = \{\text{sum, product}\}, \mathcal{P} = \{\text{SquareNumber}\}$ ar(sum) = ar(product) = 2, ar(SquareNumber) = 1 $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with $\mathbf{I} = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$ $\mathbf{I} = u_0$ $\mathbf{I} = u_0$ $\mathbf{I} = u_1$ $\mathbf{I} = \sup^{\mathcal{I}} (u_i, u_j) = u_{(i+j) \mod 7}$ for all $i, j \in \{0, \dots, 6\}$	 Semantics: Informally Example: (∀x(Block(x) → Red(x)) ∧ Block(a)) "For all objects x: if x is a block, then x is red. Also, the object called a is a block." Terms are interpreted as objects. Unary predicates denote properties of objects (to be a block, to be red, to be a square number,). General predicates denote relations between objects (to be someone's child, to have a common divisor,). Universally quantified formulas ("∀") are true
Interpretations and Variable Assignments: Example Example signature: $S = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{V} = \{x, y, z\}$, $C = \{\text{zero, one}\}, \mathcal{F} = \{\text{sum, product}\}, \mathcal{P} = \{\text{SquareNumber}\}$ ar(sum) = ar(product) = 2, ar(SquareNumber) = 1 $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$ $\text{zero}^{\mathcal{I}} = u_0$ $\text{one}^{\mathcal{I}} = u_1$ $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \mod 7} \text{ for all } i, j \in \{0, \dots, 6\}$ $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i-j) \mod 7} \text{ for all } i, j \in \{0, \dots, 6\}$	 Semantics: Informally Example: (∀x(Block(x) → Red(x)) ∧ Block(a)) "For all objects x: if x is a block, then x is red. Also, the object called a is a block." Terms are interpreted as objects. Unary predicates denote properties of objects (to be a block, to be red, to be a square number,). General predicates denote relations between objects (to be someone's child, to have a common divisor,). Universally quantified formulas ("∀") are true if they hold for every object in the universe. Existentially quantified formulas ("∃") are true
Interpretations and Variable Assignments: Example Example signature: $S = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\text{zero, one}\}, \mathcal{F} = \{\text{sum, product}\}, \mathcal{P} = \{\text{SquareNumber}\}$ ar(sum) = ar(product) = 2, ar(SquareNumber) = 1 $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with $\mathbf{U} = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$ $\mathbf{V} = \text{zero}^{\mathcal{I}} = u_0$ $\mathbf{V} = \text{one}^{\mathcal{I}} = u_1$ $\mathbf{U} = \text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \mod 7}$ for all $i, j \in \{0, \dots, 6\}$	 Semantics: Informally Example: (∀x(Block(x) → Red(x)) ∧ Block(a)) "For all objects x: if x is a block, then x is red. Also, the object called a is a block." Terms are interpreted as objects. Unary predicates denote properties of objects (to be a block, to be red, to be a square number,). General predicates denote relations between objects (to be someone's child, to have a common divisor,). Universally quantified formulas ("∀") are true if they hold for every object in the universe.

Semantics of Predicate Logic

D5. Syntax and Semantics of Predicate Logic

Semantics of Predicate Logic

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Semantics of Predicate Logic

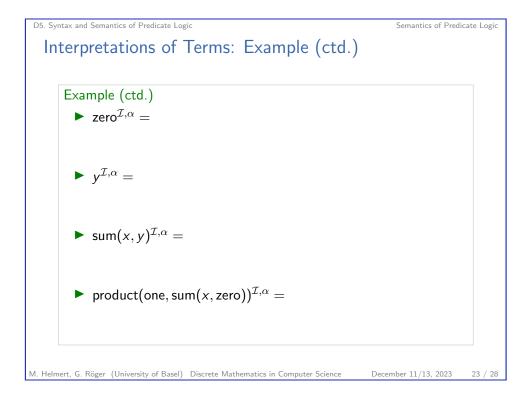
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Interpretations of Terms

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation of a Term) Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for \mathcal{S} , and let α be a variable assignment for \mathcal{S} and universe U. Let t be a term over \mathcal{S} . The interpretation of t under \mathcal{I} and α , written as $t^{\mathcal{I},\alpha}$, is the element of the universe U defined as follows: If t = x with $x \in \mathcal{V}$ (t is a variable term): $x^{\mathcal{I},\alpha} = \alpha(x)$ If t = c with $c \in \mathcal{C}$ (t is a constant term): $c^{\mathcal{I},\alpha} = c^{\mathcal{I}}$ If $t = f(t_1, \dots, t_k)$ (t is a function term): $f(t_1, \dots, t_k)^{\mathcal{I},\alpha} = f^{\mathcal{I}}(t_1^{\mathcal{I},\alpha}, \dots, t_k^{\mathcal{I},\alpha})$

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Semantics of Predicate Logic

Interpretations of Terms: Example

Example

signature: $S = \langle V, C, F, P \rangle$ with $V = \{x, y, z\}$, $C = \{\text{zero, one}\}$, $F = \{\text{sum, product}\}$, ar(sum) = ar(product) = 2

 $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle \text{ with}$ $\mathbf{\vdash} U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$ $\mathbf{\vdash} \text{ zero}^{\mathcal{I}} = u_0$ $\mathbf{\vdash} \text{ one}^{\mathcal{I}} = u_1$ $\mathbf{\vdash} \text{ sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \mod 7} \text{ for all } i, j \in \{0, \dots, 6\}$ $\mathbf{\vdash} \text{ product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \mod 7} \text{ for all } i, j \in \{0, \dots, 6\}$ $\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

D5. Syntax and Semantics of Predicate Logic Semantics of Predicate Logic Semantics of Predicate Logic Formulas Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature. Definition (Formula is Satisfied or True) Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for \mathcal{S} , and let α be a variable assignment for S and universe U. We say that \mathcal{I} and α satisfy a predicate logic formula φ (also: φ is true under \mathcal{I} and α), written: $\mathcal{I}, \alpha \models \varphi$, according to the following inductive rules: $\mathcal{I}, \alpha \models \mathsf{P}(t_1, \ldots, t_k) \quad \text{iff } \langle t_1^{\mathcal{I}, \alpha}, \ldots, t_k^{\mathcal{I}, \alpha} \rangle \in \mathsf{P}^{\mathcal{I}}$ $\mathcal{I}, \alpha \models (t_1 = t_2) \quad \text{iff} \ t_1^{\mathcal{I}, \alpha} = t_2^{\mathcal{I}, \alpha}$ $\mathcal{I}, \alpha \models \neg \varphi \quad \text{iff } \mathcal{I}, \alpha \not\models \varphi$ $\mathcal{I}, \alpha \models (\varphi \land \psi)$ iff $\mathcal{I}, \alpha \models \varphi$ and $\mathcal{I}, \alpha \models \psi$ $\mathcal{I}, \alpha \models (\varphi \lor \psi) \quad \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi$ German: \mathcal{I} und α erfüllen φ (also: φ ist wahr unter \mathcal{I} und α)

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Semantics of Predicate Logic

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Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

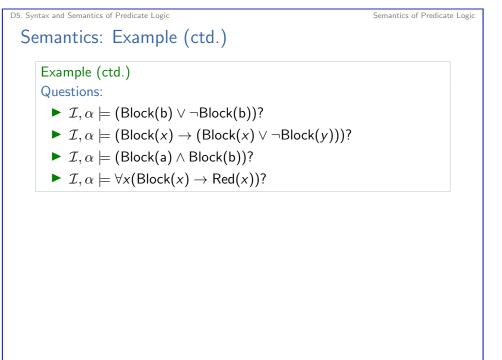
Definition (Formula is Satisfied or True)

 $\begin{array}{ll} \mathcal{I}, \alpha \models \forall x \varphi & \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for all } u \in U \\ \mathcal{I}, \alpha \models \exists x \varphi & \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for at least one } u \in U \end{array}$

where $\alpha[x := u]$ is the same variable assignment as α , except that it maps variable x to the value u. Formally:

 $(\alpha[x := u])(z) = \begin{cases} u & \text{if } z = x \\ \alpha(z) & \text{if } z \neq x \end{cases}$

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D5. Syntax and Semantics of Predicate Logic

Semantics of Predicate Logic

Semantics: Example

Example

signature: $S = \langle V, C, F, P \rangle$ with $V = \{x, y, z\}$, $C = \{a, b\}$, $F = \emptyset$, $P = \{Block, Red\}$, ar(Block) = ar(Red) = 1.

