

Discrete Mathematics in Computer Science

D4. Inference

Malte Helmert, Gabriele Röger

University of Basel

December 11, 2023

Inference Rules and Calculi

Inference: Motivation

- up to now: proof of **logical consequence** with **semantic arguments**
- no general algorithm
- **solution**: produce formulas that are logical consequences of given formulas with **syntactic inference rules**
- **advantage**: **mechanical method** that can easily be implemented as an algorithm

Inference Rules

- **Inference rules** have the form

$$\frac{\varphi_1, \dots, \varphi_k}{\psi}.$$

- Meaning: “Every model of $\varphi_1, \dots, \varphi_k$ is a model of ψ .”
- An **axiom** is an inference rule with $k = 0$.
- A set of inference rules is called a **calculus** or **proof system**.

German: Inferenzregel, Axiom, (der) Kalkül, Beweissystem

Some Inference Rules for Propositional Logic

Modus ponens $\frac{\varphi, (\varphi \rightarrow \psi)}{\psi}$

Modus tollens $\frac{\neg\psi, (\varphi \rightarrow \psi)}{\neg\varphi}$

\wedge -elimination $\frac{(\varphi \wedge \psi)}{\varphi} \quad \frac{(\varphi \wedge \psi)}{\psi}$

\wedge -introduction $\frac{\varphi, \psi}{(\varphi \wedge \psi)}$

\vee -introduction $\frac{\varphi}{(\varphi \vee \psi)}$

\leftrightarrow -elimination $\frac{(\varphi \leftrightarrow \psi)}{(\varphi \rightarrow \psi)} \quad \frac{(\varphi \leftrightarrow \psi)}{(\psi \rightarrow \varphi)}$

Derivation

Definition (Derivation)

A **derivation** or **proof** of a formula φ from a knowledge base KB is a sequence of formulas ψ_1, \dots, ψ_k with

- $\psi_k = \varphi$ and
- for all $i \in \{1, \dots, k\}$:
 - $\psi_i \in \text{KB}$, or
 - ψ_i is the result of the application of an inference rule to elements from $\{\psi_1, \dots, \psi_{i-1}\}$.

German: Ableitung, Beweis

Derivation: Example

Example

Given: $KB = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

Derivation: Example

Example

Given: $KB = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

① P (KB)

Derivation: Example

Example

Given: $KB = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

- 1 P (KB)
- 2 $(P \rightarrow Q)$ (KB)

Derivation: Example

Example

Given: $KB = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

- 1 P (KB)
- 2 $(P \rightarrow Q)$ (KB)
- 3 Q (1, 2, Modus ponens)

Derivation: Example

Example

Given: $KB = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

- 1 P (KB)
- 2 $(P \rightarrow Q)$ (KB)
- 3 Q (1, 2, Modus ponens)
- 4 $(P \rightarrow R)$ (KB)

Derivation: Example

Example

Given: $KB = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

- 1 P (KB)
- 2 $(P \rightarrow Q)$ (KB)
- 3 Q (1, 2, Modus ponens)
- 4 $(P \rightarrow R)$ (KB)
- 5 R (1, 4, Modus ponens)

Derivation: Example

Example

Given: $KB = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

- 1 P (KB)
- 2 $(P \rightarrow Q)$ (KB)
- 3 Q (1, 2, Modus ponens)
- 4 $(P \rightarrow R)$ (KB)
- 5 R (1, 4, Modus ponens)
- 6 $(Q \wedge R)$ (3, 5, \wedge -introduction)

Derivation: Example

Example

Given: $KB = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

- 1 P (KB)
- 2 $(P \rightarrow Q)$ (KB)
- 3 Q (1, 2, Modus ponens)
- 4 $(P \rightarrow R)$ (KB)
- 5 R (1, 4, Modus ponens)
- 6 $(Q \wedge R)$ (3, 5, \wedge -introduction)
- 7 $((Q \wedge R) \rightarrow S)$ (KB)

Derivation: Example

Example

Given: $KB = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

- 1 P (KB)
- 2 $(P \rightarrow Q)$ (KB)
- 3 Q (1, 2, Modus ponens)
- 4 $(P \rightarrow R)$ (KB)
- 5 R (1, 4, Modus ponens)
- 6 $(Q \wedge R)$ (3, 5, \wedge -introduction)
- 7 $((Q \wedge R) \rightarrow S)$ (KB)
- 8 S (6, 7, Modus ponens)

Derivation: Example

Example

Given: $KB = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

- 1 P (KB)
- 2 $(P \rightarrow Q)$ (KB)
- 3 Q (1, 2, Modus ponens)
- 4 $(P \rightarrow R)$ (KB)
- 5 R (1, 4, Modus ponens)
- 6 $(Q \wedge R)$ (3, 5, \wedge -introduction)
- 7 $((Q \wedge R) \rightarrow S)$ (KB)
- 8 S (6, 7, Modus ponens)
- 9 $(S \wedge R)$ (8, 5, \wedge -introduction)

Correctness and Completeness

Definition (Correctness and Completeness of a Calculus)

We write $\text{KB} \vdash_C \varphi$ if there is a derivation of φ from KB in calculus C .

(If calculus C is clear from context, also only $\text{KB} \vdash \varphi$.)

A calculus C is **correct** if for all KB and φ
 $\text{KB} \vdash_C \varphi$ implies $\text{KB} \models \varphi$.

A calculus C is **complete** if for all KB and φ
 $\text{KB} \models \varphi$ implies $\text{KB} \vdash_C \varphi$.

Correctness and Completeness

Definition (Correctness and Completeness of a Calculus)

We write $\text{KB} \vdash_C \varphi$ if there is a derivation of φ from KB in calculus C .

(If calculus C is clear from context, also only $\text{KB} \vdash \varphi$.)

A calculus C is **correct** if for all KB and φ
 $\text{KB} \vdash_C \varphi$ implies $\text{KB} \models \varphi$.

A calculus C is **complete** if for all KB and φ
 $\text{KB} \models \varphi$ implies $\text{KB} \vdash_C \varphi$.

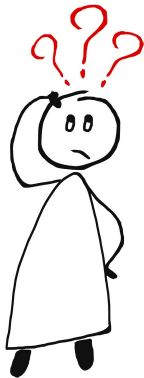
Consider calculus C , consisting of the derivation rules seen earlier.

Question: Is C correct?

Question: Is C complete?

German: korrekt, vollständig

Questions



Questions?

Summary

Summary (Consequence and Inference)

- **knowledge base**: set of formulas describing given information; satisfiable, valid etc. used like for individual formulas
- **logical consequence** $KB \models \varphi$ means that φ is true whenever (= in all models where) KB is true
- A **logical consequence** $KB \models \varphi$ allows to conclude that KB implies φ based on the semantics.
- A correct **calculus** supports such conclusions on the basis of **purely syntactical derivations** $KB \vdash \varphi$.

Further Topics

There are many aspects of propositional logic that we do not cover in this course.

- **resolution**: a commonly used proof system for formulas in CNF
- other proof systems, for example **tableaux proofs**
- algorithms for **model construction**, such as the Davis-Putnam-Logemann-Loveland (DPLL) algorithm.

↪ Foundations of AI course