Discrete Mathematics in Computer Science

D3. Normal Forms and Logical Consequence

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D3.1 Simplified Notation

Parentheses

Associativity:

$$((\varphi \land \psi) \land \chi) \equiv (\varphi \land (\psi \land \chi))$$
$$((\varphi \lor \psi) \lor \chi) \equiv (\varphi \lor (\psi \lor \chi))$$

- Placement of parentheses for a conjunction of conjunctions does not influence whether an interpretation is a model.
- ditto for disjunctions of disjunctions

→ can omit parentheses and treat this as if parentheses placed arbitrarily

- Example: $(A_1 \land A_2 \land A_3 \land A_4)$ instead of $((A_1 \land (A_2 \land A_3)) \land A_4)$
- Example: $(\neg A \lor (B \land C) \lor D)$ instead of $((\neg A \lor (B \land C)) \lor D)$

Parentheses

Does this mean we can always omit all parentheses and assume an arbitrary placement? $\rightarrow$ No!

$$((\varphi \land \psi) \lor \chi) \not\equiv (\varphi \land (\psi \lor \chi))$$

What should $\varphi \land \psi \lor \chi$ mean?

Placement of Parentheses by Convention

Often parentheses can be dropped in specific cases
and an implicit placement is assumed:

- $\neg$ binds more strongly than $\wedge$
- $\wedge$ binds more strongly than $\vee$
- $\vee$ binds more strongly than $\rightarrow$ or $\leftrightarrow$

$\rightarrow$ cf. PEMDAS/ "Punkt vor Strich"

Example

$A \vee \neg C \wedge B \rightarrow A \vee \neg D$ stands for $((A \vee (\neg C \wedge B)) \rightarrow (A \vee \neg D))$

- often harder to read
- error-prone

$\rightarrow$ not used in this course

Short Notations for Conjunctions and Disjunctions

Short notation for addition:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$\sum_{x \in \{x_1, \ldots, x_n\}} x = x_1 + x_2 + \cdots + x_n$$

Analogously:

$$\bigwedge_{i=1}^{n} \varphi_i = (\varphi_1 \land \varphi_2 \land \cdots \land \varphi_n)$$

$$\bigvee_{i=1}^{n} \varphi_i = (\varphi_1 \lor \varphi_2 \lor \cdots \lor \varphi_n)$$

$$\bigwedge_{\varphi \in X} \varphi = (\varphi_1 \land \varphi_2 \land \cdots \land \varphi_n)$$

$$\bigvee_{\varphi \in X} \varphi = (\varphi_1 \lor \varphi_2 \lor \cdots \lor \varphi_n)$$

for $X = \{\varphi_1, \ldots, \varphi_n\}$

Short Notation: Corner Cases

Is $\mathcal{I} \models \psi$ true for

$$\psi = \bigwedge_{\varphi \in X} \varphi \text{ and } \psi = \bigvee_{\varphi \in X} \varphi$$

if $X = \emptyset$ or $X = \{\chi\}$?

convention:

- $\bigwedge_{\varphi \in \emptyset} \varphi$ is a tautology.
- $\bigvee_{\varphi \in \emptyset} \varphi$ is unsatisfiable.
- $\bigwedge_{\varphi \in \{\chi\}} \varphi = \bigvee_{\varphi \in \{\chi\}} \varphi = \chi$

$\rightsquigarrow$ Why?

Exercise

Express $\bigwedge_{i=1}^{2} \bigvee_{j=1}^{3} \varphi_{ij}$ without $\bigwedge$ and $\bigvee$.

D3.2 Normal Forms

Why Normal Forms?

- A normal form is a representation with certain syntactic restrictions.
- condition for reasonable normal form: every formula must have a logically equivalent formula in normal form
- advantages:
  - can restrict proofs to formulas in normal form
  - can define algorithms only for formulas in normal form

German: Normalform

Literals, Clauses and Monomials

- A literal is an atomic proposition or the negation of an atomic proposition (e. g., A and $\neg A$).
- A clause is a disjunction of literals (e. g., $(Q \lor \neg P \lor \neg S \lor R)$).
- A monomial is a conjunction of literals (e. g., $(Q \land \neg P \land \neg S \land R)$).

The terms clause and monomial are also used for the corner case with only one literal.

German: Literal, Klausel, Monom

Terminology: Examples

Examples

- $(\neg Q \land R)$ is a monomial
- $(P \lor \neg Q)$ is a clause
- $((P \lor \neg Q) \land P)$ is neither literal nor clause nor monomial
- $\neg P$ is a literal, a clause and a monomial
- $(P \rightarrow Q)$ is neither literal nor clause nor monomial
  (but $(\neg P \lor Q)$ is a clause!)
- $(P \lor P)$ is a clause, but not a literal or monomial
- $\neg\neg P$ is neither literal nor clause nor monomial

Conjunctive Normal Form

Definition (Conjunctive Normal Form)

A formula is in conjunctive normal form (CNF) if it is a conjunction of clauses, i. e., if it has the form

$$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{m_i} L_{ij}$$

with $n, m_i > 0$ (for $1 \leq i \leq n$), where the L_{ij} are literals.

German: konjunktive Normalform (KNF)

Example

$((\neg P \lor Q) \land R \land (P \lor \neg S))$ is in CNF.

Disjunctive Normal Form

Definition (Disjunctive Normal Form)

A formula is in disjunctive normal form (DNF) if it is a disjunction of monomials, i. e., if it has the form

$$\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m_i} L_{ij}$$

with $n, m_i > 0$ (for $1 \leq i \leq n$), where the L_{ij} are literals.

German: disjunktive Normalform (DNF)

Example

$((\neg P \land Q) \lor R \lor (P \land \neg S))$ is in DNF.

CNF and DNF: Examples

Which of the following formulas are in CNF? Which are in DNF?

- $((P \vee \neg Q) \wedge P)$
- $((R \vee Q) \wedge P \wedge (R \vee S))$
- $(P \vee (\neg Q \wedge R))$
- $((P \vee \neg Q) \rightarrow P)$
- P

Construction of CNF (and DNF)

Algorithm to Construct CNF

1. Replace abbreviations $\rightarrow$ and $\leftrightarrow$ by their definitions (($\rightarrow$)-elimination and ($\leftrightarrow$)-elimination).
   $\rightsquigarrow$ formula structure: only $\lor$, $\land$, $\neg$
2. Move negations inside using De Morgan and double negation.
   $\rightsquigarrow$ formula structure: only $\lor$, $\land$, literals
3. Distribute $\lor$ over $\land$ with distributivity (strictly speaking also with commutativity).
   $\rightsquigarrow$ formula structure: CNF
4. optionally: Simplify the formula at the end or at intermediate steps (e. g., with idempotence).

Note: For DNF, distribute $\land$ over $\lor$ instead.

Constructing CNF: Example

Construction of Conjunctive Normal Form

Given: $\varphi = (((P \land \neg Q) \lor R) \rightarrow (P \lor \neg(S \lor T)))$

$$
\begin{aligned}
\varphi &\equiv (\neg((P \land \neg Q) \lor R) \lor P \lor \neg(S \lor T)) && \text{[Step 1]} \\
&\equiv ((\neg(P \land \neg Q) \land \neg R) \lor P \lor \neg(S \lor T)) && \text{[Step 2]} \\
&\equiv (((\neg P \lor \neg\neg Q) \land \neg R) \lor P \lor \neg(S \lor T)) && \text{[Step 2]} \\
&\equiv (((\neg P \lor Q) \land \neg R) \lor P \lor \neg(S \lor T)) && \text{[Step 2]} \\
&\equiv (((\neg P \lor Q) \land \neg R) \lor P \lor (\neg S \land \neg T)) && \text{[Step 2]} \\
&\equiv ((\neg P \lor Q \lor P \lor (\neg S \land \neg T)) \land \\
&\qquad (\neg R \lor P \lor (\neg S \land \neg T))) && \text{[Step 3]} \\
&\equiv (\neg R \lor P \lor (\neg S \land \neg T)) && \text{[Step 4]} \\
&\equiv ((\neg R \lor P \lor \neg S) \land (\neg R \lor P \lor \neg T)) && \text{[Step 3]}
\end{aligned}
$$

Construct DNF: Example

Construction of Disjunctive Normal Form

Given: $\varphi = (((P \land \neg Q) \lor R) \to (P \lor \neg(S \lor T)))$

$$
\begin{aligned}
\varphi &\equiv (\neg((P \land \neg Q) \lor R) \lor P \lor \neg(S \lor T)) && \text{[Step 1]}\\
&\equiv ((\neg(P \land \neg Q) \land \neg R) \lor P \lor \neg(S \lor T)) && \text{[Step 2]}\\
&\equiv (((\neg P \lor \neg\neg Q) \land \neg R) \lor P \lor \neg(S \lor T)) && \text{[Step 2]}\\
&\equiv (((\neg P \lor Q) \land \neg R) \lor P \lor \neg(S \lor T)) && \text{[Step 2]}\\
&\equiv (((\neg P \lor Q) \land \neg R) \lor P \lor (\neg S \land \neg T)) && \text{[Step 2]}\\
&\equiv ((\neg P \land \neg R) \lor (Q \land \neg R) \lor P \lor (\neg S \land \neg T)) && \text{[Step 3]}
\end{aligned}
$$

Existence of an Equivalent Formula in Normal Form

Theorem

For every formula φ there is a logically equivalent formula in CNF and a logically equivalent formula in DNF.

- "There is a" always means "there is at least one".
  Otherwise we would write "there is exactly one".
- Intuition: algorithm to construct normal form works
  with any given formula and only uses equivalence rewriting.
- actual proof would use induction over structure of formula

Size of Normal Forms

- In the worst case, a logically equivalent formula in CNF or DNF can be exponentially larger than the original formula.
- Example: for $(x_1 \vee y_1) \wedge \dots \wedge (x_n \vee y_n)$ there is no smaller logically equivalent formula in DNF than:

$$\bigvee_{S \in \mathcal{P}(\{1,\dots,n\})} \left(\bigwedge_{i \in S} x_i \wedge \bigwedge_{i \in \{1,\dots,n\} \setminus S} y_i \right)$$

- As a consequence, the construction of the CNF/DNF formula can take exponential time.

More Theorems

Theorem

A formula in CNF is a tautology iff every clause is a tautology.

Theorem

A formula in DNF is satisfiable iff at least one of its monomials is satisfiable.

$\rightsquigarrow$ both proved easily with semantics of propositional logic

D3.3 Knowledge Bases

Knowledge Bases: Example

If not DrinkBeer, then EatFish.
If EatFish and DrinkBeer,
then not EatIceCream.
If EatIceCream or not DrinkBeer,
then not EatFish.

$$\begin{aligned} \text{KB} = \{&(\neg\text{DrinkBeer} \rightarrow \text{EatFish}),\\ &((\text{EatFish} \land \text{DrinkBeer}) \rightarrow \neg\text{EatIceCream}),\\ &((\text{EatIceCream} \lor \neg\text{DrinkBeer}) \rightarrow \neg\text{EatFish})\} \end{aligned}$$

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Models for Sets of Formulas

Definition (Model for Knowledge Base)

Let KB be a knowledge base over A,
i. e., a set of propositional formulas over A.

A truth assignment $\mathcal{I}$ for A is a model for KB (written: $\mathcal{I} \models \text{KB}$) if $\mathcal{I}$ is a model for every formula $\varphi \in \text{KB}$.

German: Wissensbasis, Modell

Properties of Sets of Formulas

A knowledge base KB is

- satisfiable if KB has at least one model
- unsatisfiable if KB is not satisfiable
- valid (or a tautology) if every interpretation is a model for KB
- falsifiable if KB is no tautology

German: erfüllbar, unerfüllbar, gültig, gültig/eine Tautologie, falsifizierbar

Example I

Which of the properties does $\mathsf{KB} = \{(\mathsf{A} \land \neg\mathsf{B}), \neg(\mathsf{B} \lor \mathsf{A})\}$ have?

KB is unsatisfiable:
For every model $\mathcal{I}$ with $\mathcal{I} \models (\mathsf{A} \land \neg\mathsf{B})$ we have $\mathcal{I}(\mathsf{A}) = 1$.
This means $\mathcal{I} \models (\mathsf{B} \lor \mathsf{A})$ and thus $\mathcal{I} \not\models \neg(\mathsf{B} \lor \mathsf{A})$.

This directly implies that KB is falsifiable, not satisfiable and no tautology.

Example II

Which of the properties does

$$\mathrm{KB} = \{(\neg\mathrm{DrinkBeer} \rightarrow \mathrm{EatFish}),\\ ((\mathrm{EatFish} \land \mathrm{DrinkBeer}) \rightarrow \neg\mathrm{EatIceCream}),\\ ((\mathrm{EatIceCream} \lor \neg\mathrm{DrinkBeer}) \rightarrow \neg\mathrm{EatFish})\} \text{ have?}$$

- satisfiable, e. g. with
  $\mathcal{I} = \{\mathrm{EatFish} \mapsto 1, \mathrm{DrinkBeer} \mapsto 1, \mathrm{EatIceCream} \mapsto 0\}$
- thus not unsatisfiable
- falsifiable, e. g. with
  $\mathcal{I} = \{\mathrm{EatFish} \mapsto 0, \mathrm{DrinkBeer} \mapsto 0, \mathrm{EatIceCream} \mapsto 1\}$
- thus not valid

D3.4 Logical Consequences

Logical Consequences: Motivation

What's the secret of your long life?

I am on a strict diet: If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

Claim: the woman drinks beer to every meal.

How can we prove this?

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut/FreeDigitalPhotos.net

Logical Consequences

Definition (Logical Consequence)

Let KB be a set of formulas and φ a formula.

We say that KB logically implies φ (written as $\text{KB} \models \varphi$) if all models of KB are also models of φ.

also: KB logically entails φ, φ logically follows from KB, φ is a logical consequence of KB

German: KB impliziert φ logisch, φ folgt logisch aus KB, φ ist logische Konsequenz von KB

Attention: the symbol $\models$ is "overloaded": $\text{KB} \models \varphi$ vs. $\mathcal{I} \models \varphi$.

What if KB is unsatisfiable or the empty set?

Logical Consequences: Example

Let $\varphi = \mathsf{DrinkBeer}$ and

$$\begin{aligned}\mathsf{KB} = \{&(\neg\mathsf{DrinkBeer} \rightarrow \mathsf{EatFish}),\\ &((\mathsf{EatFish} \land \mathsf{DrinkBeer}) \rightarrow \neg\mathsf{EatIceCream}),\\ &((\mathsf{EatIceCream} \lor \neg\mathsf{DrinkBeer}) \rightarrow \neg\mathsf{EatFish})\}.\end{aligned}$$

Show: $\mathsf{KB} \models \varphi$

Proof sketch.

Proof by contradiction: assume $\mathcal{I} \models \mathsf{KB}$, but $\mathcal{I} \not\models \mathsf{DrinkBeer}$.
Then it follows that $\mathcal{I} \models \neg\mathsf{DrinkBeer}$.
Because $\mathcal{I}$ is a model of KB, we also have
$\mathcal{I} \models (\neg\mathsf{DrinkBeer} \rightarrow \mathsf{EatFish})$ and thus $\mathcal{I} \models \mathsf{EatFish}$. (Why?)
With an analogous argumentation starting from
$\mathcal{I} \models ((\mathsf{EatIceCream} \lor \neg\mathsf{DrinkBeer}) \rightarrow \neg\mathsf{EatFish})$
we get $\mathcal{I} \models \neg\mathsf{EatFish}$ and thus $\mathcal{I} \not\models \mathsf{EatFish}$. $\rightsquigarrow$ Contradiction!

Important Theorems about Logical Consequences

Theorem (Deduction Theorem)

$\mathrm{KB} \cup \{\varphi\} \models \psi$ *iff* $\mathrm{KB} \models (\varphi \rightarrow \psi)$

German: Deduktionssatz

Theorem (Contraposition Theorem)

$\mathrm{KB} \cup \{\varphi\} \models \neg\psi$ *iff* $\mathrm{KB} \cup \{\psi\} \models \neg\varphi$

German: Kontrapositionssatz

Theorem (Contradiction Theorem)

$\mathrm{KB} \cup \{\varphi\}$ *is unsatisfiable iff* $\mathrm{KB} \models \neg\varphi$

German: Widerlegungssatz

(without proof)