Discrete Mathematics in Computer Science

D2. Properties of Formulas and Equivalences

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Properties of Propositional Formulas

The Story So Far

- propositional logic based on atomic propositions
- syntax: which formulas are well-formed?
- semantics: when is a formula true?
- interpretations: important basis of semantics

Reminder: Syntax of Propositional Logic

Definition (Syntax of Propositional Logic)

Let A be a set of atomic propositions. The set of propositional formulas (over A) is inductively defined as follows:

- Every atom $a \in A$ is a propositional formula over A.
- If φ is a propositional formula over A, then so is its negation $\neg\varphi$.
- If φ and ψ are propositional formulas over A, then so is the conjunction $(\varphi \wedge \psi)$.
- If φ and ψ are propositional formulas over A, then so is the disjunction $(\varphi \vee \psi)$.

The implication $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$.
The biconditional $(\varphi \leftrightarrow \psi)$ is an abbrev. for $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.

Reminder: Semantics of Propositional Logic

Definition (Semantics of Propositional Logic)

A truth assignment (or interpretation) for a set of atomic propositions A is a function $\mathcal{I} : A \to \{0, 1\}$.

A propositional formula φ (over A) holds under $\mathcal{I}$ (written as $\mathcal{I} \models \varphi$) according to the following definition:

$$
\begin{array}{lll l}
\mathcal{I} \models a & \text{iff} & \mathcal{I}(a) = 1 & \text{(for } a \in A\text{)} \\
\mathcal{I} \models \neg\varphi & \text{iff} & \text{not } \mathcal{I} \models \varphi & \\
\mathcal{I} \models (\varphi \land \psi) & \text{iff} & \mathcal{I} \models \varphi \text{ and } \mathcal{I} \models \psi & \\
\mathcal{I} \models (\varphi \lor \psi) & \text{iff} & \mathcal{I} \models \varphi \text{ or } \mathcal{I} \models \psi &
\end{array}
$$

Properties of Propositional Formulas

A propositional formula φ is

- satisfiable if φ has at least one model
- unsatisfiable if φ is not satisfiable
- valid (or a tautology) if φ is true under every interpretation
- falsifiable if φ is no tautology

German: erfüllbar, unerfüllbar, allgemeingültig/eine Tautologie, falsifizierbar

Examples

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- Show that $(\mathsf{A} \wedge \mathsf{B})$ is not unsatisfiable.

Examples

How can we show that a formula has one of these properties?

- Show that $(A \land B)$ is satisfiable.
  $\mathcal{I} = \{A \mapsto 1, B \mapsto 1\}$ (+ simple proof that $\mathcal{I} \models (A \land B)$)
- Show that $(A \land B)$ is falsifiable.
  $\mathcal{I} = \{A \mapsto 0, B \mapsto 1\}$ (+ simple proof that $\mathcal{I} \not\models (A \land B)$)
- Show that $(A \land B)$ is not valid.
  Follows directly from falsifiability.
- Show that $(A \land B)$ is not unsatisfiable.
  Follows directly from satisfiability.

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- Show that $(A \land B)$ is falsifiable.
  $\mathcal{I} = \{A \mapsto 0, B \mapsto 1\}$ (+ simple proof that $\mathcal{I} \not\models (A \land B)$)
- Show that $(A \land B)$ is not valid.
  Follows directly from falsifiability.
- Show that $(A \land B)$ is not unsatisfiable.
  Follows directly from satisfiability.

So far all proofs by specifying one interpretation.

How to prove that a given formula is valid/unsatisfiable/
not satisfiable/not falsifiable?

$\rightsquigarrow$ must consider all possible interpretations

Truth Tables

Evaluate for all possible interpretations
if they are models of the considered formula.

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1	

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0	0	
0	1	
1	0	
1	1	

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0	Yes
1	No

$\mathcal{I}(A)$	$\mathcal{I}(B)$	$\mathcal{I} \models (A \wedge B)$
0	0	No
0	1	No
1	0	No
1	1	Yes

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1	No

$\mathcal{I}(\mathsf{A})$	$\mathcal{I}(\mathsf{B})$	$\mathcal{I} \models (\mathsf{A} \wedge \mathsf{B})$
0	0	No
0	1	No
1	0	No
1	1	Yes

$\mathcal{I}(\mathsf{A})$	$\mathcal{I}(\mathsf{B})$	$\mathcal{I} \models (\mathsf{A} \vee \mathsf{B})$
0	0	No
0	1	Yes
1	0	Yes
1	1	Yes

Truth Tables in General

Similarly in the case where we consider a formula whose building blocks are themselves arbitrary unspecified formulas:

$\mathcal{I} \models \varphi$	$\mathcal{I} \models \psi$	$\mathcal{I} \models (\varphi \wedge \psi)$
No	No	No
No	Yes	No
Yes	No	No
Yes	Yes	Yes

Truth Tables for Properties of Formulas

Is $\varphi = ((\mathsf{A} \rightarrow \mathsf{B}) \vee (\neg\mathsf{B} \rightarrow \mathsf{A}))$ valid, unsatisfiable, ...?

$\mathcal{I}(\mathsf{A})$	$\mathcal{I}(\mathsf{B})$	$\mathcal{I} \models \neg\mathsf{B}$	$\mathcal{I} \models (\mathsf{A} \rightarrow \mathsf{B})$	$\mathcal{I} \models (\neg\mathsf{B} \rightarrow \mathsf{A})$	$\mathcal{I} \models \varphi$
0	0	Yes	Yes	No	Yes
0	1	No	Yes	Yes	Yes
1	0	Yes	No	Yes	Yes
1	1	No	Yes	Yes	Yes

Connection Between Formula Properties and Truth Tables

A propositional formula φ is

- satisfiable if φ has at least one model
  $\rightsquigarrow$ result in at least one row is "Yes"
- unsatisfiable if φ is not satisfiable
  $\rightsquigarrow$ result in all rows is "No"
- valid (or a tautology) if φ is true under every interpretation
  $\rightsquigarrow$ result in all rows is "Yes"
- falsifiable if φ is no tautology
  $\rightsquigarrow$ result in at least one row is "No"

Main Disadvantage of Truth Tables

How big is a truth table with n atomic propositions?

1	2 interpretations (rows)
2	4 interpretations (rows)
3	8 interpretations (rows)
n	??? interpretations

Main Disadvantage of Truth Tables

How big is a truth table with n atomic propositions?

1	2 interpretations (rows)
2	4 interpretations (rows)
3	8 interpretations (rows)
n	2^n interpretations

Some examples: $2^{10} = 1024$, $2^{20} = 1048576$, $2^{30} = 1073741824$

$\rightsquigarrow$ not viable for larger formulas; we need a different solution

- more on difficulty of satisfiability etc.:
  Theory of Computer Science course
- practical algorithms: Foundations of AI course

Equivalences

Equivalent Formulas

Definition (Equivalence of Propositional Formulas)

Two propositional formulas φ and ψ over A are (logically) equivalent ($\varphi \equiv \psi$) if for all interpretations $\mathcal{I}$ for A it is true that $\mathcal{I} \models \varphi$ if and only if $\mathcal{I} \models \psi$.

German: logisch äquivalent

Equivalent Formulas: Example

$$((\varphi \vee \psi) \vee \chi) \equiv (\varphi \vee (\psi \vee \chi))$$

$\mathcal{I} \models \varphi$	$\mathcal{I} \models \psi$	$\mathcal{I} \models \chi$	$\mathcal{I} \models (\varphi \vee \psi)$	$\mathcal{I} \models (\psi \vee \chi)$	$\mathcal{I} \models ((\varphi \vee \psi) \vee \chi)$	$\mathcal{I} \models (\varphi \vee (\psi \vee \chi))$
No	No	No	No	No	No	No
No	No	Yes	No	Yes	Yes	Yes
No	Yes	No	Yes	Yes	Yes	Yes
No	Yes	Yes	Yes	Yes	Yes	Yes
Yes	No	No	Yes	No	Yes	Yes
Yes	No	Yes	Yes	Yes	Yes	Yes
Yes	Yes	No	Yes	Yes	Yes	Yes
Yes	Yes	Yes	Yes	Yes	Yes	Yes

Some Equivalences (1)

$$(\varphi \land \varphi) \equiv \varphi$$
$$(\varphi \lor \varphi) \equiv \varphi \qquad \text{(idempotence)}$$

German: Idempotenz

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$$(\varphi \land \varphi) \equiv \varphi$$
$$(\varphi \lor \varphi) \equiv \varphi \qquad \text{(idempotence)}$$

$$(\varphi \land \psi) \equiv (\psi \land \varphi)$$
$$(\varphi \lor \psi) \equiv (\psi \lor \varphi) \qquad \text{(commutativity)}$$

German: Idempotenz, Kommutativität

Some Equivalences (1)

$$
\begin{aligned}
(\varphi \land \varphi) &\equiv \varphi \\
(\varphi \lor \varphi) &\equiv \varphi && \text{(idempotence)} \\[1ex]
(\varphi \land \psi) &\equiv (\psi \land \varphi) \\
(\varphi \lor \psi) &\equiv (\psi \lor \varphi) && \text{(commutativity)} \\[1ex]
((\varphi \land \psi) \land \chi) &\equiv (\varphi \land (\psi \land \chi)) \\
((\varphi \lor \psi) \lor \chi) &\equiv (\varphi \lor (\psi \lor \chi)) && \text{(associativity)}
\end{aligned}
$$

German: Idempotenz, Kommutativität, Assoziativität

Some Equivalences (2)

$$(\varphi \wedge (\varphi \vee \psi)) \equiv \varphi$$
$$(\varphi \vee (\varphi \wedge \psi)) \equiv \varphi \qquad \text{(absorption)}$$

German: Absorption

Some Equivalences (2)

$$(\varphi \land (\varphi \lor \psi)) \equiv \varphi$$

$$(\varphi \lor (\varphi \land \psi)) \equiv \varphi \qquad \text{(absorption)}$$

$$(\varphi \land (\psi \lor \chi)) \equiv ((\varphi \land \psi) \lor (\varphi \land \chi))$$

$$(\varphi \lor (\psi \land \chi)) \equiv ((\varphi \lor \psi) \land (\varphi \lor \chi)) \qquad \text{(distributivity)}$$

German: Absorption, Distributivität

Some Equivalences (3)

$$\neg\neg\varphi \equiv \varphi \qquad \text{(double negation)}$$

German: Doppelnegation

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$$\neg\neg\varphi \equiv \varphi \qquad \text{(double negation)}$$

$$\neg(\varphi \land \psi) \equiv (\neg\varphi \lor \neg\psi)$$
$$\neg(\varphi \lor \psi) \equiv (\neg\varphi \land \neg\psi) \qquad \text{(De Morgan's rules)}$$

German: Doppelnegation, De Morgansche Regeln

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$$(\varphi \lor \psi) \equiv \varphi \text{ if } \varphi \text{ tautology}$$

$$(\varphi \land \psi) \equiv \psi \text{ if } \varphi \text{ tautology} \qquad \text{(tautology rules)}$$

German: Doppelnegation, De Morgansche Regeln, Tautologieregeln

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$$(\varphi \lor \psi) \equiv \varphi \text{ if } \varphi \text{ tautology}$$
$$(\varphi \land \psi) \equiv \psi \text{ if } \varphi \text{ tautology} \qquad \text{(tautology rules)}$$

$$(\varphi \lor \psi) \equiv \psi \text{ if } \varphi \text{ unsatisfiable}$$
$$(\varphi \land \psi) \equiv \varphi \text{ if } \varphi \text{ unsatisfiable} \qquad \text{(unsatisfiability rules)}$$

German: Doppelnegation, De Morgansche Regeln, Tautologieregeln, Unerfüllbarkeitsregeln

Substitution Theorem

Theorem (Substitution Theorem)

Let φ and φ' be equivalent propositional formulas over A.
Let ψ be a propositional formula with (at least) one occurrence of the subformula φ.

Then ψ is equivalent to ψ', where ψ' is constructed from ψ by replacing an occurrence of φ in ψ with φ'.

German: Ersetzbarkeitstheorem

(without proof)

Application of Equivalences: Example

$$(P \land (Q \lor \neg P)) \equiv ((P \land Q) \lor (P \land \neg P)) \qquad \text{(distributivity)}$$

Application of Equivalences: Example

$$
\begin{aligned}
(\mathsf{P} \land (\mathsf{Q} \lor \lnot \mathsf{P})) &\equiv ((\mathsf{P} \land \mathsf{Q}) \lor (\mathsf{P} \land \lnot \mathsf{P})) && \text{(distributivity)} \\
&\equiv ((\mathsf{P} \land \lnot \mathsf{P}) \lor (\mathsf{P} \land \mathsf{Q})) && \text{(commutativity)}
\end{aligned}
$$

Application of Equivalences: Example

$$
\begin{aligned}
(P \land (Q \lor \neg P)) &\equiv ((P \land Q) \lor (P \land \neg P)) && \text{(distributivity)} \\
&\equiv ((P \land \neg P) \lor (P \land Q)) && \text{(commutativity)} \\
&\equiv (P \land Q) && \text{(unsatisfiability rule)}
\end{aligned}
$$