Discrete Mathematics in Computer Science

C2. Paths and Connectivity

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Walks, Paths, Tours and Cycles

Traversing Graphs

- When dealing with graphs, we are often not just interested in the neighbours, but also in the neighbours of neighbours, the neighbours of neighbours of neighbours, etc.
- Similarly, for digraphs we often want to follow longer chains of successors (or chains of predecessors).

Traversing Graphs

- When dealing with graphs, we are often not just interested in the neighbours, but also in the neighbours of neighbours, the neighbours of neighbours of neighbours, etc.
- Similarly, for digraphs we often want to follow longer chains of successors (or chains of predecessors).

Examples:

- circuits: follow predecessors of signals to identify possible causes of faulty signals
- pathfinding: follow edges/arcs to find paths
- control flow graphs: follow arcs to identify dead code
- computer networks: determine if part of the network is unreachable

Walks

Definition (Walk)

A walk of length n in a graph (V, E) is a tuple $\langle v_0, v_1, \ldots, v_n \rangle \in V^{n+1}$ s.t. $\{v_i, v_{i+1}\} \in E$ for all $0 \leq i < n$.

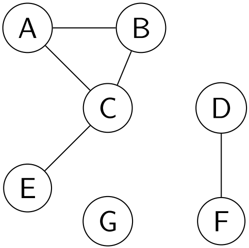
A walk of length n in a digraph (N, A) is a tuple $\langle v_0, v_1, \ldots, v_n \rangle \in N^{n+1}$ s.t. $(v_i, v_{i+1}) \in A$ for all $0 \leq i < n$.

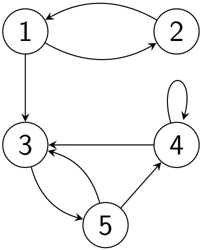
German: Wanderung

Notes:

- The length of the walk does not equal the length of the tuple!
- The case $n = 0$ is allowed.
- Vertices may repeat along a walk.

Walks – Example





examples of walks:

- $\langle B, C, A \rangle$
- $\langle B, C, A, B \rangle$
- $\langle D, F, D \rangle$
- $\langle B, A, B, C, E \rangle$
- $\langle B \rangle$

examples of walks:

- $\langle 4, 4, 4, 4 \rangle$
- $\langle 3, 5, 3, 5 \rangle$
- $\langle 2, 1, 3 \rangle$
- $\langle 4 \rangle$
- $\langle 4, 4 \rangle$

Walks – Terminology

Definition

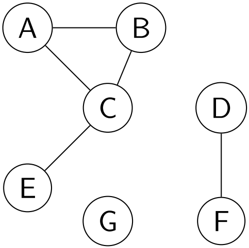
Let $\pi = \langle v_0, \dots, v_n \rangle$ be a walk in a graph or digraph G.

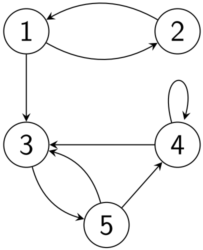
- We say π is a walk from v_0 to v_n.
- A walk with $v_i \neq v_j$ for all $0 \leq i < j \leq n$ is called a path.
- A walk of length 0 is called an empty walk/path.
- A walk with $v_0 = v_n$ is called a tour.
- A tour with $n \geq 1$ (digraphs) or $n \geq 3$ (graphs) and $v_i \neq v_j$ for all $1 \leq i < j \leq n$ is called a cycle.

German: von/nach, Pfad, leer, Tour, Zyklus

Note: Terminology is not very consistent in the literature.

Walks, Paths, Tours, Cycles – Example





Which walks are paths, tours, cycles?

- $\langle B, C, A \rangle$
- $\langle B, C, A, B \rangle$
- $\langle D, F, D \rangle$
- $\langle B, A, B, C, E \rangle$
- $\langle B \rangle$
- $\langle 4, 4, 4, 4 \rangle$
- $\langle 3, 5, 3, 5 \rangle$
- $\langle 2, 1, 3 \rangle$
- $\langle 4 \rangle$
- $\langle 4, 4 \rangle$

Reachability

Reachability

Definition (successor and reachability)

Let G be a graph (digraph).
The successor relation S_G and reachability relation R_G are relations over the vertices/nodes of G defined as follows:

- $(u, v) \in \mathrm{S}_G$ iff $\{u, v\}$ is an edge ((u, v) is an arc) of G
- $(u, v) \in \mathrm{R}_G$ iff there exists a walk from u to v

If $(u, v) \in \mathrm{R}_G$, we say that v is reachable from u.

German: Nachfolger-/Erreichbarkeitsrelation, erreichbar

Reachability as Closure

Recall the n-fold composition R^n of a relation R over set S:

- $R^1 = R$
- $R^{n+1} = R \circ R^n$

also: $R^0 = \{(x, x) \mid x \in S\}$ (0-fold composition is identity relation)

Theorem

Let G be a graph or digraph. Then:
$(u, v) \in \mathrm{S}_G^n$ iff there exists a walk of length n from u to v.

Corollary

Let G be a graph or digraph. Then $\mathrm{R}_G = \bigcup_{n=0}^{\infty} \mathrm{S}_G^n$.

In other words, the reachability relation is the reflexive and transitive closure of the successor relation.

Reachability as Closure – Proof (1)

Proof.

To simplify notation, we assume $G = (N, A)$ is a digraph.
Graphs are analogous.
Proof by induction over n.

...

Reachability as Closure – Proof (1)

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Graphs are analogous.
Proof by induction over n.

induction base ($n = 0$):
By definition of the 0-fold composition, we have $(u, v) \in \mathsf{S}_G^0$ iff $u = v$, and a walk of length 0 from u to v exists iff $u = v$.
Hence, the two conditions are equivalent. ...

Reachability as Closure – Proof (2)

Proof (continued).

induction step ($n \to n+1$):

□

Reachability as Closure – Proof (2)

Proof (continued).

induction step ($n \rightarrow n+1$):

($\Rightarrow$) : Let $(u, v) \in \mathsf{S}_G^{n+1}$.
By definition of R^{n+1}, we get $(u, v) \in \mathsf{S}_G \circ \mathsf{S}_G^n$.
By definition of $\circ$ there exists w with $(u, w) \in \mathsf{S}_G^n$ and $(w, v) \in \mathsf{S}_G$.
From the induction hypothesis, there exists a length-n walk $\langle x_0, \ldots, x_n \rangle$ with $x_0 = u$ and $x_n = w$.
Then $\langle x_0, \ldots, x_n, v \rangle$ is a length-$(n+1)$ walk from u to v.

□

Reachability as Closure – Proof (2)

Proof (continued).

induction step ($n \to n+1$):

$(\Rightarrow)$: Let $(u, v) \in \mathsf{S}_G^{n+1}$.
By definition of R^{n+1}, we get $(u, v) \in \mathsf{S}_G \circ \mathsf{S}_G^n$.
By definition of $\circ$ there exists w with $(u, w) \in \mathsf{S}_G^n$ and $(w, v) \in \mathsf{S}_G$.
From the induction hypothesis, there exists a length-n walk $\langle x_0, \ldots, x_n \rangle$ with $x_0 = u$ and $x_n = w$.
Then $\langle x_0, \ldots, x_n, v \rangle$ is a length-$(n+1)$ walk from u to v.

$(\Leftarrow)$: Let $\langle x_0, \ldots, x_{n+1} \rangle$ be a length-$(n+1)$ walk from u to v ($x_0 = u$, $x_{n+1} = v$). Then $(x_n, x_{n+1}) = (x_n, v) \in A$.
Also, $\langle x_0, \ldots, x_n \rangle$ is a length-n walk from x_0 to x_n.
From the IH we get $(u, x_n) = (x_0, x_n) \in \mathsf{S}_G^n$.
Together with $(x_n, v) \in \mathsf{S}_G$ this shows $(u, v) \in \mathsf{S}_G \circ \mathsf{S}_G^n = \mathsf{S}_G^{n+1}$.

□

Connected Components

Overview

- In this section, we study reachability of graphs in more depth.
- We show that it makes no difference whether we define reachability in terms of walks or paths, and that reachability in graphs is an equivalence relation.
- This leads to the connected components of a graph.
- In digraphs, reachability is not always an equivalence relation.
- However, we can define two variants of reachability that give rise to weakly or strongly connected components.

Walks vs. Paths

Theorem

Let G be a graph or digraph.
There exists a path from u to v iff there exists a walk from u to v.

In other words, there is a path from u to v iff v is reachable from u.

□

Walks vs. Paths

Theorem

Let G be a graph or digraph.
There exists a path from u to v iff there exists a walk from u to v.

In other words, there is a path from u to v iff v is reachable from u.

Proof.

($\Rightarrow$): obvious because paths are special cases of walks

($\Leftarrow$): Proof by contradiction. Assume there exist u, v such that there exists a walk from u to v, but no path. Let $\pi = \langle w_0, \ldots, w_n \rangle$ be such a counterexample walk of minimal length.
Because π is not a path, some vertex/node must repeat.
Select i and j with $i < j$ and $w_i = w_j$.
Then $\pi' = \langle w_0, \ldots, w_i, w_{j+1}, \ldots, w_n \rangle$ also is a walk from u to v.
If π' is a path, we have a contradiction.
If not, it is a shorter counterexample: also a contradiction. □

Reachability in Graphs is an Equivalence Relation

Theorem

For every graph G, the reachability relation R_G is an equivalence relation.

In directed graphs, this result does not hold (easy to see).

Proof.

We already know reachability is reflexive and transitive.
To prove symmetry:

$$
\begin{aligned}
& (u, v) \in \mathrm{R}_G \\
\Rightarrow\ & \text{there is a walk } \langle w_0, \ldots, w_n \rangle \text{ from } u \text{ to } v \\
\Rightarrow\ & \langle w_n, \ldots, w_0 \rangle \text{ is a walk from } v \text{ to } u \\
\Rightarrow\ & (v, u) \in \mathrm{R}_G
\end{aligned}
$$

□

Connected Components

Definition (connected components, connected)

In a graph G, the equivalence classes
of the reachability relation of G
are called the connected components of G.

A graph is called connected if it has at most 1
connected component.

German: Zusammenhangskomponenten, zusammenhängend

Remark: The graph $(\emptyset, \emptyset)$ has 0 connected components.
It is the only such graph.

Weakly Connected Components

Definition (weakly connected components, weakly connected)

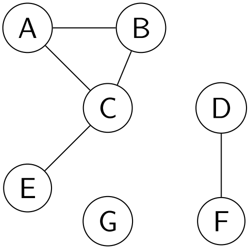
In a digraph G, the equivalence classes
of the reachability relation of the induced graph of G
are called the weakly connected components of G.

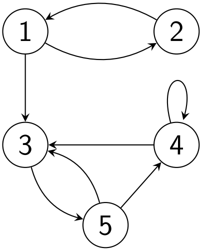
A digraph is called weakly connected if it has at most 1
weakly connected component.

German: schwache Zshk., schwach zusammenhängend

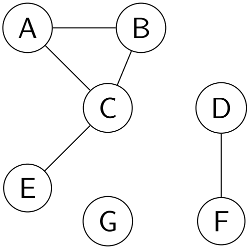
Remark: The digraph $(\emptyset, \emptyset)$ has 0 weakly connected components.
It is the only such digraph.

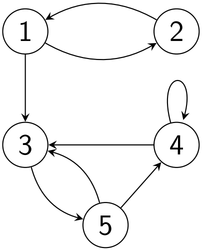
(Weakly) Connected Components – Example





(Weakly) Connected Components – Example





connected components:

- $\{A, B, C, E\}$
- $\{D, F\}$
- $\{G\}$

weakly connected components:

- $\{1, 2, 3, 4, 5\}$

Mutual Reachability

Definition (mutually reachable)

Let G be a graph or digraph.
Vertices/nodes u and v in G are called mutually reachable if v is reachable from u and u is reachable from v.
We write M_G for the mutual reachability relation of G

German: gegenseitig erreichbar

Note: In graphs, $\mathrm{M}_G = \mathrm{R}_G$. (Why?)

Mutual Reachability is an Equivalence Relation

Theorem

For every digraph G, the mutual reachability relation M_G is an equivalence relation.

Mutual Reachability is an Equivalence Relation

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Proof.

Note that $(u, v) \in \mathsf{M}_G$ iff $(u, v) \in \mathsf{R}_G$ and $(v, u) \in \mathsf{R}_G$.

- reflexivity: for all v, we have $(v, v) \in \mathsf{M}_G$ because $(v, v) \in \mathsf{R}_G$

□

Mutual Reachability is an Equivalence Relation

Theorem

For every digraph G, the mutual reachability relation M_G is an equivalence relation.

Proof.

Note that $(u, v) \in \mathsf{M}_G$ iff $(u, v) \in \mathsf{R}_G$ and $(v, u) \in \mathsf{R}_G$.

- reflexivity: for all v, we have $(v, v) \in \mathsf{M}_G$ because $(v, v) \in \mathsf{R}_G$
- symmetry: Let $(u, v) \in \mathsf{M}_G$. Then $(v, u) \in \mathsf{M}_G$ is obvious.

□

Mutual Reachability is an Equivalence Relation

Theorem

For every digraph G, the mutual reachability relation M_G is an equivalence relation.

Proof.

Note that $(u, v) \in \mathrm{M}_G$ iff $(u, v) \in \mathrm{R}_G$ and $(v, u) \in \mathrm{R}_G$.

- reflexivity: for all v, we have $(v, v) \in \mathrm{M}_G$ because $(v, v) \in \mathrm{R}_G$
- symmetry: Let $(u, v) \in \mathrm{M}_G$. Then $(v, u) \in \mathrm{M}_G$ is obvious.
- transitivity: Let $(u, v) \in \mathrm{M}_G$ and $(v, w) \in \mathrm{M}_G$.
  Then: $(u, v) \in \mathrm{R}_G$, $(v, u) \in \mathrm{R}_G$, $(v, w) \in \mathrm{R}_G$, $(w, v) \in \mathrm{R}_G$.
  Transitivity of R_G yields $(u, w) \in \mathrm{R}_G$ and $(w, u) \in \mathrm{R}_G$,
  and hence $(u, w) \in \mathrm{M}_G$.

□

Strongly Connected Components

Definition (strongly connected components, strongly connected)

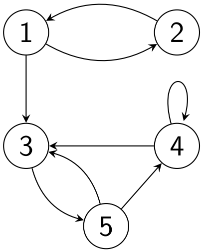
In a digraph G, the equivalence classes
of the mutual reachability relation
are called the strongly connected components of G.

A digraph is called strongly connected if it has at most 1
strongly connected component.

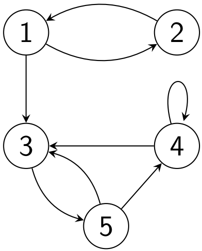
German: starke Zshk., stark zusammenhängend

Remark: The digraph $(\emptyset, \emptyset)$ has 0 strongly connected components.
It is the only such digraph.

Strongly Connected Components – Example



Strongly Connected Components – Example



strongly connected components:

- $\{1, 2\}$
- $\{3, 4, 5\}$