

# Discrete Mathematics in Computer Science

## C2. Paths and Connectivity

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C2.1 Walks, Paths, Tours and Cycles

C2.2 Reachability

C2.3 Connected Components

# C2.1 Walks, Paths, Tours and Cycles

# Traversing Graphs

- ▶ When dealing with graphs, we are often not just interested in the neighbours, but also in the **neighbours of neighbours**, the **neighbours of neighbours of neighbours**, etc.
- ▶ Similarly, for digraphs we often want to follow longer chains of successors (or chains of predecessors).

## Examples:

- ▶ circuits: follow predecessors of signals to identify possible causes of faulty signals
- ▶ pathfinding: follow edges/arcs to find paths
- ▶ control flow graphs: follow arcs to identify dead code
- ▶ computer networks: determine if part of the network is unreachable

# Walks

## Definition (Walk)

A **walk** of **length**  $n$  in a graph  $(V, E)$  is a tuple  $\langle v_0, v_1, \dots, v_n \rangle \in V^{n+1}$  s.t.  $\{v_i, v_{i+1}\} \in E$  for all  $0 \leq i < n$ .

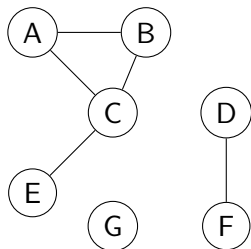
A **walk** of **length**  $n$  in a digraph  $(N, A)$  is a tuple  $\langle v_0, v_1, \dots, v_n \rangle \in N^{n+1}$  s.t.  $(v_i, v_{i+1}) \in A$  for all  $0 \leq i < n$ .

German: Wanderung

## Notes:

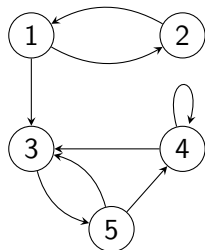
- ▶ The length of the walk does not equal the length of the tuple!
- ▶ The case  $n = 0$  is allowed.
- ▶ Vertices may repeat along a walk.

# Walks – Example



examples of walks:

- ▶  $\langle B, C, A \rangle$
- ▶  $\langle B, C, A, B \rangle$
- ▶  $\langle D, F, D \rangle$
- ▶  $\langle B, A, B, C, E \rangle$
- ▶  $\langle B \rangle$



examples of walks:

- ▶  $\langle 4, 4, 4, 4 \rangle$
- ▶  $\langle 3, 5, 3, 5 \rangle$
- ▶  $\langle 2, 1, 3 \rangle$
- ▶  $\langle 4 \rangle$
- ▶  $\langle 4, 4 \rangle$

# Walks – Terminology

## Definition

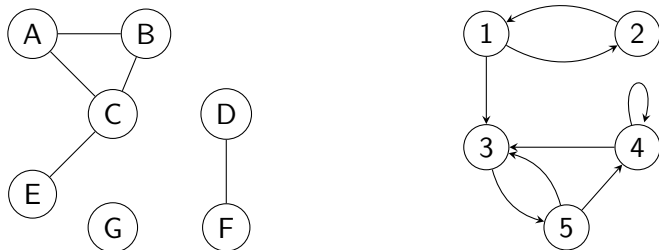
Let  $\pi = \langle v_0, \dots, v_n \rangle$  be a walk in a graph or digraph  $G$ .

- ▶ We say  $\pi$  is a walk **from**  $v_0$  **to**  $v_n$ .
- ▶ A walk with  $v_i \neq v_j$  for all  $0 \leq i < j \leq n$  is called a **path**.
- ▶ A walk of length 0 is called an **empty** walk/path.
- ▶ A walk with  $v_0 = v_n$  is called a **tour**.
- ▶ A tour with  $n \geq 1$  (digraphs) or  $n \geq 3$  (graphs) and  $v_i \neq v_j$  for all  $1 \leq i < j \leq n$  is called a **cycle**.

**German:** von/nach, Pfad, leer, Tour, Zyklus

**Note:** Terminology is not very consistent in the literature.

# Walks, Paths, Tours, Cycles – Example



Which walks are paths, tours, cycles?

- ▶  $\langle B, C, A \rangle$
- ▶  $\langle B, C, A, B \rangle$
- ▶  $\langle D, F, D \rangle$
- ▶  $\langle B, A, B, C, E \rangle$
- ▶  $\langle B \rangle$
- ▶  $\langle 4, 4, 4, 4 \rangle$
- ▶  $\langle 3, 5, 3, 5 \rangle$
- ▶  $\langle 2, 1, 3 \rangle$
- ▶  $\langle 4 \rangle$
- ▶  $\langle 4, 4 \rangle$



## C2.2 Reachability

# Reachability

## Definition (successor and reachability)

Let  $G$  be a graph (digraph).

The **successor relation**  $S_G$  and **reachability relation**  $R_G$  are relations over the vertices/nodes of  $G$  defined as follows:

- ▶  $(u, v) \in S_G$  iff  $\{u, v\}$  is an edge ( $(u, v)$  is an arc) of  $G$
- ▶  $(u, v) \in R_G$  iff there exists a walk from  $u$  to  $v$

If  $(u, v) \in R_G$ , we say that  **$v$  is reachable from  $u$** .

**German:** Nachfolger-/Erreichbarkeitsrelation, erreichbar

## Reachability as Closure

Recall the  $n$ -fold composition  $R^n$  of a relation  $R$  over set  $S$ :

▶  $R^1 = R$

▶  $R^{n+1} = R \circ R^n$

also:  $R^0 = \{(x, x) \mid x \in S\}$  (0-fold composition is identity relation)

### Theorem

*Let  $G$  be a graph or digraph. Then:*

*$(u, v) \in S_G^n$  iff there exists a walk of length  $n$  from  $u$  to  $v$ .*

### Corollary

*Let  $G$  be a graph or digraph. Then  $R_G = \bigcup_{n=0}^{\infty} S_G^n$ .*

In other words, the reachability relation is the reflexive and transitive closure of the successor relation.

# Reachability as Closure – Proof (1)

## Proof.

To simplify notation, we assume  $G = (N, A)$  is a digraph.

Graphs are analogous.

Proof by induction over  $n$ .

induction base ( $n = 0$ ):

By definition of the 0-fold composition, we have  $(u, v) \in S_G^0$  iff  $u = v$ , and a walk of length 0 from  $u$  to  $v$  exists iff  $u = v$ .

Hence, the two conditions are equivalent.

...

## Reachability as Closure – Proof (2)

Proof (continued).

induction step ( $n \rightarrow n + 1$ ):

$(\Rightarrow)$  : Let  $(u, v) \in S_G^{n+1}$ .

By definition of  $R^{n+1}$ , we get  $(u, v) \in S_G \circ S_G^n$ .

By definition of  $\circ$  there exists  $w$  with  $(u, w) \in S_G^n$  and  $(w, v) \in S_G$ .

From the induction hypothesis, there exists a length- $n$  walk  $\langle x_0, \dots, x_n \rangle$  with  $x_0 = u$  and  $x_n = w$ .

Then  $\langle x_0, \dots, x_n, v \rangle$  is a length- $(n + 1)$  walk from  $u$  to  $v$ .

$(\Leftarrow)$  : Let  $\langle x_0, \dots, x_{n+1} \rangle$  be a length- $(n + 1)$  walk from  $u$  to  $v$  ( $x_0 = u, x_{n+1} = v$ ). Then  $(x_n, x_{n+1}) = (x_n, v) \in A$ .

Also,  $\langle x_0, \dots, x_n \rangle$  is a length- $n$  walk from  $x_0$  to  $x_n$ .

From the IH we get  $(u, x_n) = (x_0, x_n) \in S_G^n$ .

Together with  $(x_n, v) \in S_G$  this shows  $(u, v) \in S_G \circ S_G^n = S_G^{n+1}$ .



## C2.3 Connected Components

# Overview

- ▶ In this section, we study reachability of graphs in more depth.
- ▶ We show that it makes no difference whether we define reachability in terms of walks or paths, and that reachability in graphs is an **equivalence relation**.
- ▶ This leads to the **connected components** of a graph.
- ▶ In digraphs, reachability is not always an equivalence relation.
- ▶ However, we can define two variants of reachability that give rise to **weakly** or **strongly connected components**.

# Walks vs. Paths

## Theorem

Let  $G$  be a graph or digraph.

There exists a path from  $u$  to  $v$  iff there exists a walk from  $u$  to  $v$ .

In other words, there is a path from  $u$  to  $v$  iff  $v$  is reachable from  $u$ .

## Proof.

( $\Rightarrow$ ): obvious because paths are special cases of walks

( $\Leftarrow$ ): Proof by contradiction. Assume there exist  $u, v$  such that there exists a walk from  $u$  to  $v$ , but no path. Let  $\pi = \langle w_0, \dots, w_n \rangle$  be such a counterexample walk of minimal length.

Because  $\pi$  is not a path, some vertex/node must repeat.

Select  $i$  and  $j$  with  $i < j$  and  $w_i = w_j$ .

Then  $\pi' = \langle w_0, \dots, w_i, w_{j+1}, \dots, w_n \rangle$  also is a walk from  $u$  to  $v$ .

If  $\pi'$  is a path, we have a contradiction.

If not, it is a shorter counterexample: also a contradiction.  $\square$



# Reachability in Graphs is an Equivalence Relation

## Theorem

For every *graph*  $G$ , the reachability relation  $R_G$  is an *equivalence relation*.

In *directed graphs*, this result does not hold (easy to see).

## Proof.

We already know reachability is reflexive and transitive.

To prove symmetry:

$$\begin{aligned}(u, v) \in R_G \\ \Rightarrow \text{there is a walk } \langle w_0, \dots, w_n \rangle \text{ from } u \text{ to } v \\ \Rightarrow \langle w_n, \dots, w_0 \rangle \text{ is a walk from } v \text{ to } u \\ \Rightarrow (v, u) \in R_G\end{aligned}$$



# Connected Components

## Definition (connected components, connected)

In a graph  $G$ , the equivalence classes of the reachability relation of  $G$  are called the **connected components** of  $G$ .

A graph is called **connected** if it has at most 1 connected component.

**German:** Zusammenhangskomponenten, zusammenhängend

**Remark:** The graph  $(\emptyset, \emptyset)$  has 0 connected components. It is the only such graph.

# Weakly Connected Components

## Definition (weakly connected components, weakly connected)

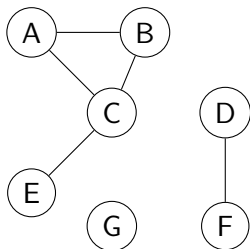
In a digraph  $G$ , the equivalence classes of the reachability relation of the induced graph of  $G$  are called the **weakly connected components** of  $G$ .

A digraph is called **weakly connected** if it has at most 1 weakly connected component.

**German:** schwache Zshk., schwach zusammenhängend

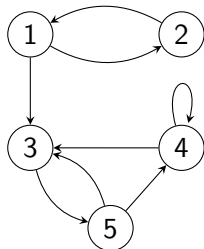
**Remark:** The digraph  $(\emptyset, \emptyset)$  has 0 weakly connected components. It is the only such digraph.

# (Weakly) Connected Components – Example



connected components:

- ▶ {A, B, C, E}
- ▶ {D, F}
- ▶ {G}



weakly connected components:

- ▶ {1, 2, 3, 4, 5}

# Mutual Reachability

## Definition (mutually reachable)

Let  $G$  be a graph or digraph.

Vertices/nodes  $u$  and  $v$  in  $G$  are called **mutually reachable** if  $v$  is reachable from  $u$  and  $u$  is reachable from  $v$ .

We write  $M_G$  for the **mutual reachability** relation of  $G$

**German:** gegenseitig erreichbar

**Note:** In graphs,  $M_G = R_G$ . (Why?)

# Mutual Reachability is an Equivalence Relation

## Theorem

For every *digraph*  $G$ , the mutual reachability relation  $M_G$  is an *equivalence relation*.

## Proof.

Note that  $(u, v) \in M_G$  iff  $(u, v) \in R_G$  and  $(v, u) \in R_G$ .

- ▶ **reflexivity:** for all  $v$ , we have  $(v, v) \in M_G$  because  $(v, v) \in R_G$
- ▶ **symmetry:** Let  $(u, v) \in M_G$ . Then  $(v, u) \in M_G$  is obvious.
- ▶ **transitivity:** Let  $(u, v) \in M_G$  and  $(v, w) \in M_G$ .  
Then:  $(u, v) \in R_G$ ,  $(v, u) \in R_G$ ,  $(v, w) \in R_G$ ,  $(w, v) \in R_G$ .  
Transitivity of  $R_G$  yields  $(u, w) \in R_G$  and  $(w, u) \in R_G$ ,  
and hence  $(u, w) \in M_G$ .



# Strongly Connected Components

**Definition (strongly connected components, strongly connected)**

In a digraph  $G$ , the equivalence classes of the mutual reachability relation

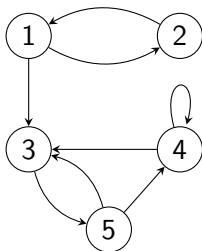
are called the **strongly connected components** of  $G$ .

A digraph is called **strongly connected** if it has at most 1 strongly connected component.

**German:** starke Zshk., stark zusammenhängend

**Remark:** The digraph  $(\emptyset, \emptyset)$  has 0 strongly connected components. It is the only such digraph.

# Strongly Connected Components – Example



strongly connected components:

- ▶  $\{1, 2\}$
- ▶  $\{3, 4, 5\}$