

# Discrete Mathematics in Computer Science

## C1. Introduction to Graphs

Malte Helmert, Gabriele Röger

University of Basel

November 6, 2023

# Graphs and Directed Graphs

# Graphs

**Graphs** (of various kinds) are ubiquitous in Computer Science and its applications.

Some examples:

- Boolean circuits in hardware design
- control flow graphs in compilers
- pathfinding in video games
- computer networks
- neural networks
- social networks

# Graph Theory

- Graph theory was founded in 1736 by Leonhard Euler's study of the Seven Bridges of Königsberg problem.
- It remains one of the main areas of discrete mathematics to this day.

## More on Euler and the Seven Bridges of Königsberg:



- The Seven Bridges of Königsberg – Numberphile.  
<https://youtu.be/W18FDEA1jRQ>

# Graphs and Directed Graphs – Definitions

## Definition (Graph)

A **graph** (also: **undirected graph**) is a pair  $G = (V, E)$ , where

- $V$  is a finite set called the set of **vertices**, and
- $E \subseteq \{\{u, v\} \subseteq V \mid u \neq v\}$  is called the set of **edges**.

**German:** Graph, ungerichteter Graph, Knoten, Kanten

# Graphs and Directed Graphs – Definitions

## Definition (Graph)

A **graph** (also: **undirected graph**) is a pair  $G = (V, E)$ , where

- $V$  is a finite set called the set of **vertices**, and
- $E \subseteq \{\{u, v\} \subseteq V \mid u \neq v\}$  is called the set of **edges**.

**German:** Graph, ungerichteter Graph, Knoten, Kanten

## Definition (Directed Graph)

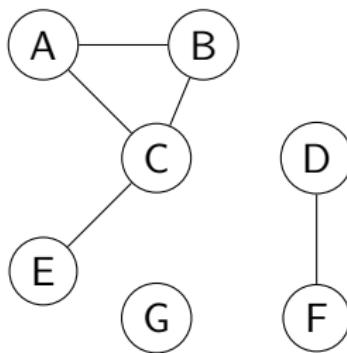
A **directed graph** (also: **digraph**) is a pair  $G = (N, A)$ , where

- $N$  is a finite set called the set of **nodes**, and
- $A \subseteq N \times N$  is called the set of **arcs**.

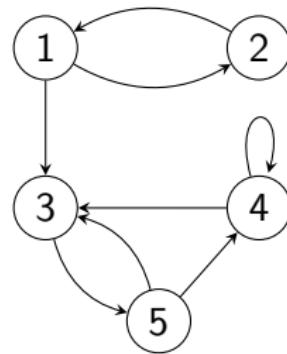
**German:** gerichteter Graph, Digraph, Knoten, Kanten/Pfeile

# Graphs and Directed Graphs – Pictorially

often described pictorially:



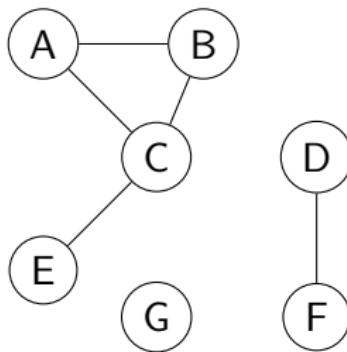
graph  $(V, E)$



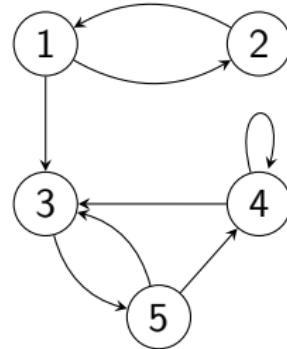
directed graph  $(N, A)$

# Graphs and Directed Graphs – Pictorially

often described pictorially:



graph  $(V, E)$



directed graph  $(N, A)$

- $V = \{A, B, C, D, E, F, G\}$
- $E = \{\{A, B\}, \{A, C\}, \{B, C\}, \{C, E\}, \{D, F\}\}$

- $N = \{1, 2, 3, 4, 5\}$
- $A = \{(1, 2), (1, 3), (2, 1), (3, 5), (4, 3), (4, 4), (5, 3), (5, 4)\}$

# Relationship to Relations

graphs vs. directed graphs:

- edges are **sets** of two elements, arcs are **pairs**
- arcs can be **self-loops**  $(v, v)$ ; edges cannot (**why not?**)

(di-)graphs vs. relations:

- A directed graph  $(N, A)$  is essentially identical to  
(= contains the same information as)  
an **arbitrary relation**  $R_A$  over the finite set  $N$ :  
 $u R_A v$  iff  $(u, v) \in A$
- A graph  $(V, E)$  is essentially identical to  
an **irreflexive symmetric** relation  $R_E$  over the finite set  $V$ :  
 $u R_E v$  iff  $\{u, v\} \in E$

## Other Kinds of Graphs

many variations exist, for example:

- self-loops may be allowed in edges (“non-simple” graphs)
- labeled graphs: additional information associated with vertices and/or edges
- weighted graphs: numbers associated with edges
- multigraphs: multiple edges between same vertices allowed
- mixed graphs: both edges and arcs allowed
- hypergraphs: edges can involve more than 2 vertices
- infinite graphs: may have infinitely many vertices/edges

# Graph Terminology

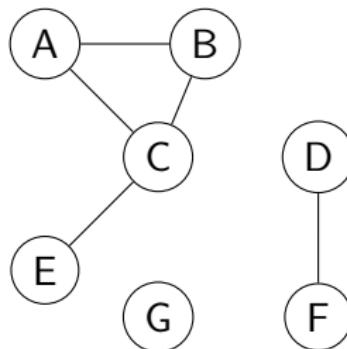
## Definition (Graph Terminology)

Let  $(V, E)$  be a graph.

- $u$  and  $v$  are the **endpoints** of the edge  $\{u, v\} \in E$
- $u$  and  $v$  are **incident** to the edge  $\{u, v\} \in E$
- $u$  and  $v$  are **adjacent** if  $\{u, v\} \in E$
- the vertices adjacent with  $v \in V$  are its **neighbours**  $\text{neigh}(v)$ :  
$$\text{neigh}(v) = \{w \in V \mid \{v, w\} \in E\}$$
- the number of neighbours of  $v \in V$  is its **degree**  $\text{deg}(v)$ :  
$$\text{deg}(v) = |\text{neigh}(v)|$$

**German:** Endknoten, inzident, adjazent/benachbart, Nachbarn, Grad

## Graph Terminology – Examples



endpoints, incident, adjacent, neighbours, degree

# Directed Graph Terminology

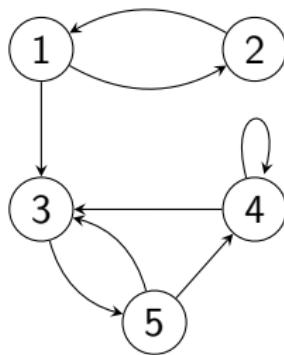
## Definition (Directed Graph Terminology)

Let  $(N, A)$  be a directed graph.

- $u$  is the **tail** and  $v$  is the **head** of the arc  $(u, v) \in A$ ;  
we say  $(u, v)$  is an arc **from  $u$  to  $v$**
- $u$  and  $v$  are **incident** to the arc  $(u, v) \in A$
- $u$  is a **predecessor** of  $v$  and  $v$  is a **successor** of  $u$  if  $(u, v) \in A$
- the predecessors and successor of  $v$  are written as  
 $\text{pred}(v) = \{u \in N \mid (u, v) \in A\}$  and  
 $\text{succ}(v) = \{w \in N \mid (v, w) \in A\}$
- the number of predecessors/successors of  $v \in N$  is its  
**indegree/outdegree**:  $\text{indeg}(v) = |\text{pred}(v)|$ ,  
 $\text{outdeg}(v) = |\text{succ}(v)|$

**German:** Fuss, Kopf, inzident, Vorgänger, Nachfolger,  
Eingangs-/Ausgangsgrad

## Directed Graph Terminology – Examples



head, tail, predecessors, successors, indegree, outdegree

# Induced Graphs and Degree Lemma

# Induced Graph of a Directed Graph

Definition (undirected graph induced by a directed graph)

Let  $G = (N, A)$  be a directed graph.

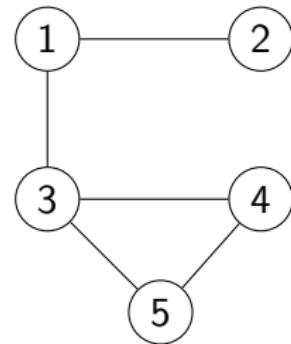
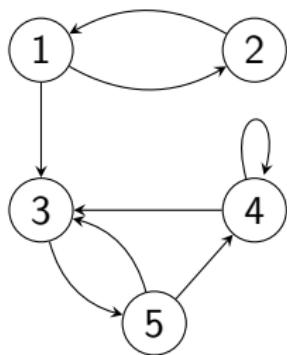
The (undirected) **graph induced by  $G$**  is the graph  $(N, E)$  with  $E = \{\{u, v\} \mid (u, v) \in A, u \neq v\}$ .

German: induziert

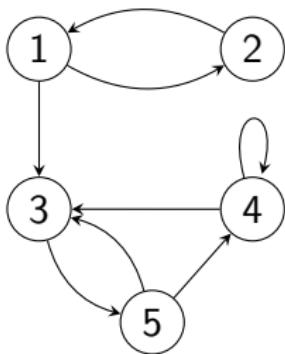
Questions:

- Why require  $u \neq v$ ?
- If  $|N| = n$  and  $|A| = m$ , how many vertices and edges does the induced graph have?
- How does the answer change if  $G$  has no self-loops?

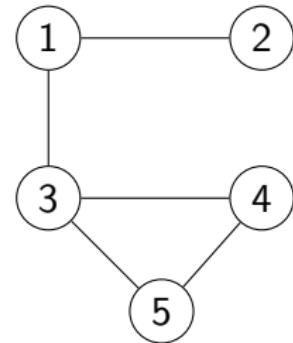
## Induced Graph of a Directed Graph – Example



## Induced Graph of a Directed Graph – Example



- $N = \{1, 2, 3, 4, 5\}$
- $A = \{(1, 2), (1, 3), (2, 1), (3, 5), (4, 3), (4, 4), (5, 3), (5, 4)\}$



- $V = \{1, 2, 3, 4, 5\}$
- $E = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$

## Degree Lemma

Lemma (degree lemma for directed graphs)

Let  $(N, A)$  be a directed graph.

Then  $\sum_{v \in N} \text{indeg}(v) = \sum_{v \in N} \text{outdeg}(v) = |A|$ .

**Intuitively:** every arc contributes 1 to the indegree of one node and 1 to the outdegree of one node.

## Degree Lemma

### Lemma (degree lemma for directed graphs)

Let  $(N, A)$  be a directed graph.

Then  $\sum_{v \in N} \text{indeg}(v) = \sum_{v \in N} \text{outdeg}(v) = |A|$ .

**Intuitively:** every arc contributes 1 to the indegree of one node and 1 to the outdegree of one node.

### Lemma (degree lemma for undirected graphs)

Let  $(V, E)$  be a graph.

Then  $\sum_{v \in V} \deg(v) = 2|E|$ .

**Intuitively:** every edge contributes 1 to the degree of two vertices.

## Degree Lemma

### Lemma (degree lemma for directed graphs)

Let  $(N, A)$  be a directed graph.

Then  $\sum_{v \in N} \text{indeg}(v) = \sum_{v \in N} \text{outdeg}(v) = |A|$ .

**Intuitively:** every arc contributes 1 to the indegree of one node and 1 to the outdegree of one node.

### Lemma (degree lemma for undirected graphs)

Let  $(V, E)$  be a graph.

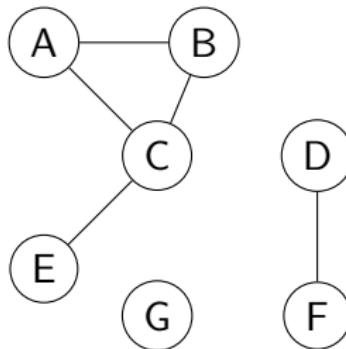
Then  $\sum_{v \in V} \deg(v) = 2|E|$ .

**Intuitively:** every edge contributes 1 to the degree of two vertices.

### Corollary

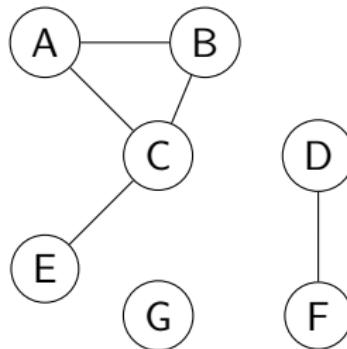
*Every graph has an even number of vertices with odd degree.*

## Degree Lemma – Example



$$\sum_{v \in V} \deg(v) = \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G)$$

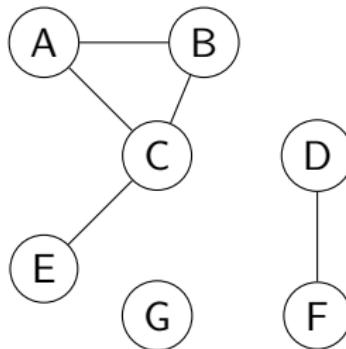
## Degree Lemma – Example



$$\sum_{v \in V} \deg(v)$$

$$\begin{aligned} &= \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G) \\ &= 2 + 2 + 3 + 1 + 1 + 1 + 0 \end{aligned}$$

## Degree Lemma – Example



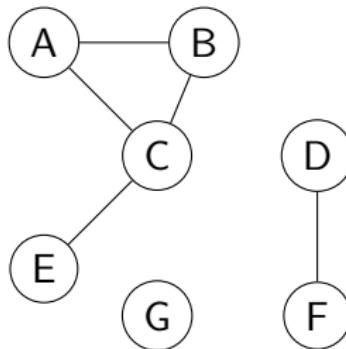
$$\sum_{v \in V} \deg(v)$$

$$= \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G)$$

$$= 2 + 2 + 3 + 1 + 1 + 1 + 0$$

$$= 10 = 2 \cdot 5 = 2|E|$$

## Degree Lemma – Example



$$\sum_{v \in V} \deg(v)$$

$$\begin{aligned} &= \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G) \\ &= 2 + 2 + 3 + 1 + 1 + 1 + 0 \\ &= 10 = 2 \cdot 5 = 2|E| \end{aligned}$$

4 vertices with odd degree

## Degree Lemma – Proof (1)

Proof of degree lemma for directed graphs.

$$\begin{aligned}\sum_{v \in N} \text{indeg}(v) &= \sum_{v \in N} |\text{pred}(v)| \\&= \sum_{v \in N} |\{u \mid u \in N, (u, v) \in A\}| \\&= \sum_{v \in N} |\{(u, v) \mid u \in N, (u, v) \in A\}| \\&= \left| \bigcup_{v \in N} \{(u, v) \mid u \in N, (u, v) \in A\} \right| \\&= |\{(u, v) \mid u \in N, v \in N, (u, v) \in A\}| \\&= |A|.\end{aligned}$$

$\sum_{v \in N} \text{outdeg}(v) = |A|$  is analogous. □

## Degree Lemma – Proof (2)

We omit the proof for undirected graphs, which can be conducted similarly.

One possible proof strategy that reuses the result we proved:

- Define **directed** graph  $(V, A)$  from the graph  $(V, E)$  by orienting each edge into an arc arbitrarily.
- Observe  $\deg(v) = \text{indeg}(v) + \text{outdeg}(v)$ , where  $\deg$  refers to the graph and  $\text{indeg}/\text{outdeg}$  to the directed graph.
- Use the degree lemma for directed graphs:  
$$\sum_{v \in V} \deg(v) = \sum_{v \in V} (\text{indeg}(v) + \text{outdeg}(v)) = \sum_{v \in V} \text{indeg}(v) + \sum_{v \in V} \text{outdeg}(v) = |A| + |A| = 2|A| = 2|E|$$