Discrete Mathematics in Computer Science

C1. Introduction to Graphs

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Graphs and Directed Graphs

Graphs

Graphs (of various kinds) are ubiquitous in Computer Science and its applications.

Some examples:

- Boolean circuits in hardware design
- control flow graphs in compilers
- pathfinding in video games
- computer networks
- neural networks
- social networks

Graph Theory

- Graph theory was founded in 1736 by Leonhard Euler's study of the Seven Bridges of Königsberg problem.
- It remains one of the main areas of discrete mathematics to this day.

More on Euler and the Seven Bridges of Königsberg:

- The Seven Bridges of Königsberg – Numberphile.
  `https://youtu.be/W18FDEA1jRQ`

Graphs and Directed Graphs – Definitions

Definition (Graph)

A graph (also: undirected graph) is a pair $G = (V, E)$, where

- V is a finite set called the set of vertices, and
- $E \subseteq \{\{u, v\} \subseteq V \mid u \neq v\}$ is called the set of edges.

German: Graph, ungerichteter Graph, Knoten, Kanten

Graphs and Directed Graphs – Definitions

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Definition (Directed Graph)

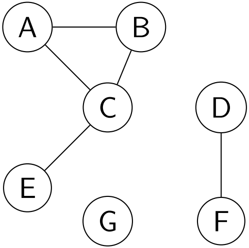
A directed graph (also: digraph) is a pair $G = (N, A)$, where

- N is a finite set called the set of nodes, and
- $A \subseteq N \times N$ is called the set of arcs.

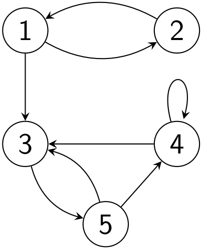
German: gerichteter Graph, Digraph, Knoten, Kanten/Pfeile

Graphs and Directed Graphs – Pictorially

often described pictorially:



graph (V, E)



directed graph (N, A)

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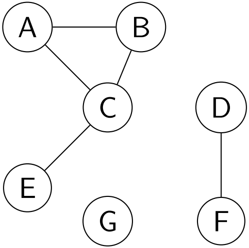

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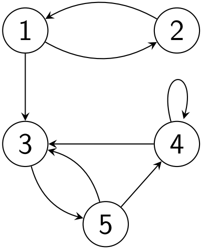


directed graph (N, A)

- $V = \{\text{A}, \text{B}, \text{C}, \text{D}, \text{E}, \text{F}, \text{G}\}$
- $E = \{\{\text{A}, \text{B}\}, \{\text{A}, \text{C}\}, \{\text{B}, \text{C}\}, \{\text{C}, \text{E}\}, \{\text{D}, \text{F}\}\}$
- $N = \{1, 2, 3, 4, 5\}$
- $A = \{(1, 2), (1, 3), (2, 1), (3, 5), (4, 3), (4, 4), (5, 3), (5, 4)\}$

Relationship to Relations

graphs vs. directed graphs:

- edges are sets of two elements, arcs are pairs
- arcs can be self-loops (v, v); edges cannot (why not?)

(di-)graphs vs. relations:

- A directed graph (N, A) is essentially identical to
  (= contains the same information as)
  an arbitrary relation R_A over the finite set N:
  $u \, R_A \, v$ iff $(u, v) \in A$
- A graph (V, E) is essentially identical to
  an irreflexive symmetric relation R_E over the finite set V:
  $u \, R_E \, v$ iff $\{u, v\} \in E$

Other Kinds of Graphs

many variations exist, for example:

- self-loops may be allowed in edges ("non-simple" graphs)
- labeled graphs: additional information associated with vertices and/or edges
- weighted graphs: numbers associated with edges
- multigraphs: multiple edges between same vertices allowed
- mixed graphs: both edges and arcs allowed
- hypergraphs: edges can involve more than 2 vertices
- infinite graphs: may have infinitely many vertices/edges

Graph Terminology

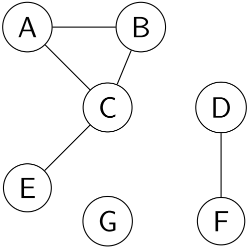
Definition (Graph Terminology)

Let (V, E) be a graph.

- u and v are the endpoints of the edge $\{u, v\} \in E$
- u and v are incident to the edge $\{u, v\} \in E$
- u and v are adjacent if $\{u, v\} \in E$
- the vertices adjacent with $v \in V$ are its neighbours $\text{neigh}(v)$: $\text{neigh}(v) = \{w \in V \mid \{v, w\} \in E\}$
- the number of neighbours of $v \in V$ is its degree $\deg(v)$: $\deg(v) = |\text{neigh}(v)|$

German: Endknoten, inzident, adjazent/benachbart, Nachbarn, Grad

Graph Terminology – Examples



endpoints, incident, adjacent, neighbours, degree

Directed Graph Terminology

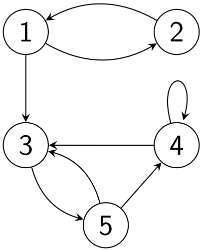
Definition (Directed Graph Terminology)

Let (N, A) be a directed graph.

- u is the tail and v is the head of the arc $(u, v) \in A$;
  we say (u, v) is an arc from u to v
- u and v are incident to the arc $(u, v) \in A$
- u is a predecessor of v and v is a successor of u if $(u, v) \in A$
- the predecessors and successor of v are written as
  $\text{pred}(v) = \{u \in N \mid (u, v) \in A\}$ and
  $\text{succ}(v) = \{w \in N \mid (v, w) \in A\}$
- the number of predecessors/successors of $v \in N$ is its
  indegree/outdegree: $\text{indeg}(v) = |\text{pred}(v)|$,
  $\text{outdeg}(v) = |\text{succ}(v)|$

German: Fuss, Kopf, inzident, Vorgänger, Nachfolger,
Eingangs-/Ausgangsgrad

Directed Graph Terminology – Examples



head, tail, predecessors, successors, indegree, outdegree

Induced Graphs and Degree Lemma

Induced Graph of a Directed Graph

Definition (undirected graph induced by a directed graph)

Let $G = (N, A)$ be a directed graph.
The (undirected) graph induced by G is the graph (N, E) with $E = \{\{u, v\} \mid (u, v) \in A, u \neq v\}$.

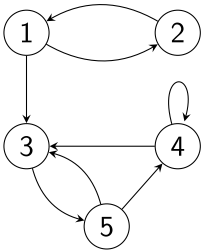
German: induziert

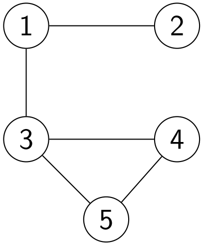
Questions:

- Why require $u \neq v$?
- If $|N| = n$ and $|A| = m$, how many vertices and edges does the induced graph have?
- How does the answer change if G has no self-loops?

Induced Graph of a Directed Graph – Example

Induced Graph of a Directed Graph – Example





- $N = \{1, 2, 3, 4, 5\}$
- $A = \{(1,2), (1,3), (2,1), (3,5), (4,3), (4,4), (5,3), (5,4)\}$

- $V = \{1, 2, 3, 4, 5\}$
- $E = \{\{1,2\}, \{1,3\}, \{3,4\}, \{3,5\}, \{4,5\}\}$

Degree Lemma

Lemma (degree lemma for directed graphs)

Let (N, A) *be a directed graph.*
Then $\sum_{v \in N} \mathrm{indeg}(v) = \sum_{v \in N} \mathrm{outdeg}(v) = |A|$.

Intuitively: every arc contributes 1 to the indegree of one node and 1 to the outdegree of one node.

Degree Lemma

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Intuitively: every arc contributes 1 to the indegree of one node and 1 to the outdegree of one node.

Lemma (degree lemma for undirected graphs)

Let (V, E) *be a graph.*
Then $\sum_{v \in V} \deg(v) = 2|E|$.

Intuitively: every edge contributes 1 to the degree of two vertices.

Degree Lemma

Lemma (degree lemma for directed graphs)

Let (N, A) be a directed graph.
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Lemma (degree lemma for undirected graphs)

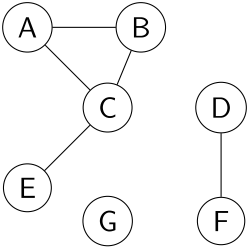
Let (V, E) be a graph.
Then $\sum_{v \in V} \deg(v) = 2|E|$.

Intuitively: every edge contributes 1 to the degree of two vertices.

Corollary

Every graph has an even number of vertices with odd degree.

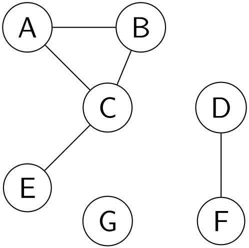
Degree Lemma – Example



$$\sum_{v \in V} \deg(v)$$

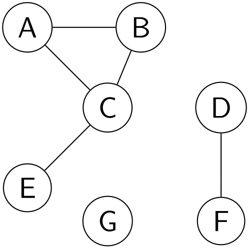
$$= \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G)$$

Degree Lemma – Example



$$\sum_{v \in V} \deg(v)$$
$$= \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G)$$
$$= 2 + 2 + 3 + 1 + 1 + 1 + 0$$

Degree Lemma – Example



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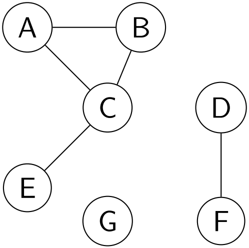
$$= 2 + 2 + 3 + 1 + 1 + 1 + 0$$
$$= 10 = 2 \cdot 5 = 2|E|$$

Degree Lemma – Example



$$\sum_{v \in V} \deg(v)$$
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$$= 2 + 2 + 3 + 1 + 1 + 1 + 0$$
$$= 10 = 2 \cdot 5 = 2|E|$$

4 vertices with odd degree

Degree Lemma – Proof (1)

Proof of degree lemma for directed graphs.

$$
\begin{aligned}
\sum_{v \in N} \mathrm{indeg}(v) &= \sum_{v \in N} |\mathrm{pred}(v)| \\
&= \sum_{v \in N} |\{u \mid u \in N, (u, v) \in A\}| \\
&= \sum_{v \in N} |\{(u, v) \mid u \in N, (u, v) \in A\}| \\
&= \left| \bigcup_{v \in N} \{(u, v) \mid u \in N, (u, v) \in A\} \right| \\
&= |\{(u, v) \mid u \in N, v \in N, (u, v) \in A\}| \\
&= |A|.
\end{aligned}
$$

$\sum_{v \in N} \mathrm{outdeg}(v) = |A|$ is analogous. □

Degree Lemma – Proof (2)

We omit the proof for undirected graphs,
which can be conducted similarly.

One possible proof strategy that reuses the result we proved:

- Define directed graph (V, A) from the graph (V, E) by orienting each edge into an arc arbitrarily.
- Observe $\deg(v) = \text{indeg}(v) + \text{outdeg}(v)$, where deg refers to the graph and indeg/outdeg to the directed graph.
- Use the degree lemma for directed graphs:
  $\sum_{v \in V} \deg(v) = \sum_{v \in V}(\text{indeg}(v) + \text{outdeg}(v)) = \sum_{v \in V} \text{indeg}(v) + \sum_{v \in V} \text{outdeg}(v) = |A| + |A| = 2|A| = 2|E|$