

Discrete Mathematics in Computer Science

C1. Introduction to Graphs

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C1.1 Graphs and Directed Graphs

C1.2 Induced Graphs and Degree Lemma

C1.1 Graphs and Directed Graphs

Graphs

Graphs (of various kinds) are ubiquitous in Computer Science and its applications.

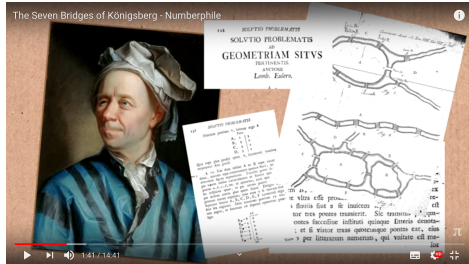
Some examples:

- ▶ Boolean circuits in hardware design
- ▶ control flow graphs in compilers
- ▶ pathfinding in video games
- ▶ computer networks
- ▶ neural networks
- ▶ social networks

Graph Theory

- ▶ **Graph theory** was founded in 1736 by Leonhard Euler's study of the **Seven Bridges of Königsberg** problem.
- ▶ It remains one of the main areas of discrete mathematics to this day.

More on Euler and the Seven Bridges of Königsberg:



- ▶ The Seven Bridges of Königsberg – Numberphile.
<https://youtu.be/W18FDEA1jRQ>

Graphs and Directed Graphs – Definitions

Definition (Graph)

A **graph** (also: **undirected graph**) is a pair $G = (V, E)$, where

- ▶ V is a finite set called the set of **vertices**, and
- ▶ $E \subseteq \{\{u, v\} \subseteq V \mid u \neq v\}$ is called the set of **edges**.

German: Graph, ungerichteter Graph, Knoten, Kanten

Definition (Directed Graph)

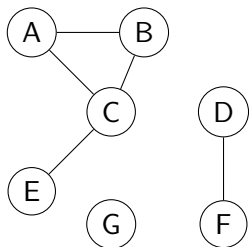
A **directed graph** (also: **digraph**) is a pair $G = (N, A)$, where

- ▶ N is a finite set called the set of **nodes**, and
- ▶ $A \subseteq N \times N$ is called the set of **arcs**.

German: gerichteter Graph, Digraph, Knoten, Kanten/Pfeile

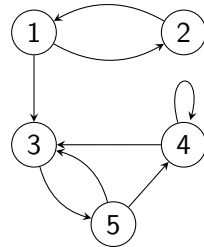
Graphs and Directed Graphs – Pictorially

often described pictorially:



graph (V, E)

- ▶ $V = \{A, B, C, D, E, F, G\}$
- ▶ $E = \{\{A, B\}, \{A, C\}, \{B, C\}, \{C, E\}, \{D, F\}\}$



directed graph (N, A)

- ▶ $N = \{1, 2, 3, 4, 5\}$
- ▶ $A = \{(1, 2), (2, 1), (1, 3), (3, 5), (4, 3), (4, 4), (5, 3), (5, 4)\}$

Relationship to Relations

graphs vs. directed graphs:

- ▶ edges are **sets** of two elements, arcs are **pairs**
- ▶ arcs can be **self-loops** (v, v) ; edges cannot (**why not?**)

(di-)graphs vs. relations:

- ▶ A directed graph (N, A) is essentially identical to (= contains the same information as) an **arbitrary relation** R_A over the finite set N :
 $u R_A v$ iff $(u, v) \in A$
- ▶ A graph (V, E) is essentially identical to an **irreflexive symmetric** relation R_E over the finite set V :
 $u R_E v$ iff $\{u, v\} \in E$

Other Kinds of Graphs

many variations exist, for example:

- ▶ self-loops may be allowed in edges (“non-simple” graphs)
- ▶ labeled graphs: additional information associated with vertices and/or edges
- ▶ weighted graphs: numbers associated with edges
- ▶ multigraphs: multiple edges between same vertices allowed
- ▶ mixed graphs: both edges and arcs allowed
- ▶ hypergraphs: edges can involve more than 2 vertices
- ▶ infinite graphs: may have infinitely many vertices/edges

Graph Terminology

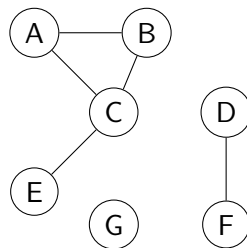
Definition (Graph Terminology)

Let (V, E) be a graph.

- ▶ u and v are the **endpoints** of the edge $\{u, v\} \in E$
- ▶ u and v are **incident** to the edge $\{u, v\} \in E$
- ▶ u and v are **adjacent** if $\{u, v\} \in E$
- ▶ the vertices adjacent with $v \in V$ are its **neighbours** $\text{neigh}(v)$:
 $\text{neigh}(v) = \{w \in V \mid \{v, w\} \in E\}$
- ▶ the number of neighbours of $v \in V$ is its **degree** $\text{deg}(v)$:
 $\text{deg}(v) = |\text{neigh}(v)|$

German: Endknoten, inzident, adjazent/benachbart, Nachbarn, Grad

Graph Terminology – Examples



endpoints, incident, adjacent, neighbours, degree

Directed Graph Terminology

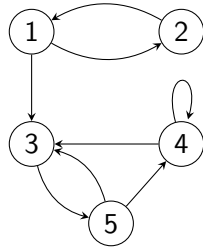
Definition (Directed Graph Terminology)

Let (N, A) be a directed graph.

- ▶ u is the **tail** and v is the **head** of the arc $(u, v) \in A$;
we say (u, v) is an arc **from** u **to** v
- ▶ u and v are **incident** to the arc $(u, v) \in A$
- ▶ u is a **predecessor** of v and v is a **successor** of u if $(u, v) \in A$
- ▶ the predecessors and successor of v are written as
 $\text{pred}(v) = \{u \in N \mid (u, v) \in A\}$ and
 $\text{succ}(v) = \{w \in N \mid (v, w) \in A\}$
- ▶ the number of predecessors/successors of $v \in N$ is its
indegree/outdegree: $\text{indeg}(v) = |\text{pred}(v)|$,
 $\text{outdeg}(v) = |\text{succ}(v)|$

German: Fuss, Kopf, inzident, Vorgänger, Nachfolger, Eingangs-/Ausgangsgrad

Directed Graph Terminology – Examples



head, tail, predecessors, successors, indegree, outdegree

C1.2 Induced Graphs and Degree Lemma

Induced Graph of a Directed Graph

Definition (undirected graph induced by a directed graph)

Let $G = (N, A)$ be a directed graph.

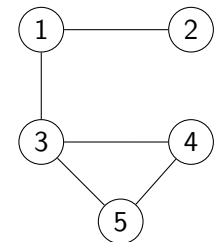
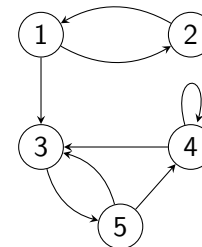
The (undirected) **graph induced by G** is the graph (N, E) with $E = \{\{u, v\} \mid (u, v) \in A, u \neq v\}$.

German: induziert

Questions:

- ▶ Why require $u \neq v$?
- ▶ If $|N| = n$ and $|A| = m$, how many vertices and edges does the induced graph have?
- ▶ How does the answer change if G has no self-loops?

Induced Graph of a Directed Graph – Example



- ▶ $N = \{1, 2, 3, 4, 5\}$
- ▶ $A = \{(1, 2), (1, 3), (2, 1), (3, 5), (4, 3), (4, 4), (5, 3), (5, 4)\}$
- ▶ $V = \{1, 2, 3, 4, 5\}$
- ▶ $E = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$

Degree Lemma

Lemma (degree lemma for directed graphs)

Let (N, A) be a directed graph.

Then $\sum_{v \in N} \text{indeg}(v) = \sum_{v \in N} \text{outdeg}(v) = |A|$.

Intuitively: every arc contributes 1 to the indegree of one node and 1 to the outdegree of one node.

Lemma (degree lemma for undirected graphs)

Let (V, E) be a graph.

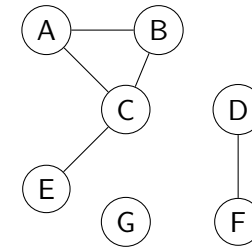
Then $\sum_{v \in V} \text{deg}(v) = 2|E|$.

Intuitively: every edge contributes 1 to the degree of two vertices.

Corollary

Every graph has an even number of vertices with odd degree.

Degree Lemma – Example



$$\begin{aligned} \sum_{v \in V} \text{deg}(v) &= \text{deg}(A) + \text{deg}(B) + \text{deg}(C) + \text{deg}(D) + \text{deg}(E) + \text{deg}(F) + \text{deg}(G) \\ &= 2 + 2 + 3 + 1 + 1 + 1 + 0 \\ &= 10 = 2 \cdot 5 = 2|E| \end{aligned}$$

4 vertices with odd degree

Degree Lemma – Proof (1)

Proof of degree lemma for directed graphs.

$$\begin{aligned} \sum_{v \in N} \text{indeg}(v) &= \sum_{v \in N} |\text{pred}(v)| \\ &= \sum_{v \in N} |\{u \mid u \in N, (u, v) \in A\}| \\ &= \sum_{v \in N} |\{(u, v) \mid u \in N, (u, v) \in A\}| \\ &= \left| \bigcup_{v \in N} \{(u, v) \mid u \in N, (u, v) \in A\} \right| \\ &= |\{(u, v) \mid u \in N, v \in N, (u, v) \in A\}| \\ &= |A|. \end{aligned}$$

$\sum_{v \in N} \text{outdeg}(v) = |A|$ is analogous. \square

Degree Lemma – Proof (2)

We omit the proof for undirected graphs, which can be conducted similarly.

One possible proof strategy that reuses the result we proved:

- ▶ Define **directed** graph (V, A) from the graph (V, E) by orienting each edge into an arc arbitrarily.
- ▶ Observe $\text{deg}(v) = \text{indeg}(v) + \text{outdeg}(v)$, where deg refers to the graph and $\text{indeg}/\text{outdeg}$ to the directed graph.
- ▶ Use the degree lemma for directed graphs:

$$\begin{aligned} \sum_{v \in V} \text{deg}(v) &= \sum_{v \in V} (\text{indeg}(v) + \text{outdeg}(v)) = \\ &= \sum_{v \in V} \text{indeg}(v) + \sum_{v \in V} \text{outdeg}(v) = |A| + |A| = 2|A| = 2|E| \end{aligned}$$