Discrete Mathematics in Computer Science

B9. Divisibility & Modular Arithmetic

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Divisibility

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- If yes then n is a multiple of m and m divides n.
- We consider a generalization of this concept to the integers.

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Definition (divisor, multiple)

Let $m, n \in \mathbb{Z}$. If there exists a $k \in \mathbb{Z}$ such that $mk = n$, we say that m divides n, m is a divisor of n or n is a multiple of m and write this as $m \mid n$.

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Which of the following are true?

- $2 \mid 4$
- $-2 \mid 4$
- $2 \mid -4$
- $4 \mid 2$
- $3 \mid 4$

Divisibility and Linear Combinations

Theorem (Linear combinations)

Let a, b and d be integers. If $d \mid a$ and $d \mid b$ then for all integers x and y it holds that $d \mid xa + yb$.

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Proof.

If $d \mid a$ and $d \mid b$ then there are $k, k' \in \mathbb{Z}$ such that $kd = a$ and $k'd = b$.
It holds for all $x, y \in \mathbb{Z}$ that $xa + yb = xkd + yk'd = (xk + yk')d$.
As x, y, k, k' are integers, $xk + yk'$ is integer, thus $d \mid xa + yb$. $\square$

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Some consequences:

- $d \mid a - b$ iff $d \mid b - a$
- If $d \mid a$ and $d \mid b$ then $d \mid a + b$ and $d \mid a - b$.
- If $d \mid a$ then $d \mid -8a$.

Multiplication and Exponentiation

Theorem

Let $a, b, c \in \mathbb{Z}$ *and* $n \in \mathbb{N}_{>0}$.
If $a \mid b$ *then* $ac \mid bc$ *and* $a^n \mid b^n$.

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From $ak = b$, we also get $b^n = (ak)^n = a^n k^n$, so $a^n \mid b^n$. □

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Proof.

- reflexivity: For all $m \in \mathbb{N}_0$ it holds that $m \cdot 1 = m$, so $m \mid m$.
- transitivity: If $m \mid n$ and $n \mid o$ there are $k, k' \in \mathbb{Z}$ such that $mk = n$ and $nk' = o$.
  With $k'' = kk'$ it holds then that $o = nk' = mkk' = mk''$, and consequently $m \mid o$.

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  (with $m \neq 0$) that $kk' = 1$.

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  Since k and k' are integers, this implies $k = k' = 1$ or $k = k' = -1$. As $mk = n$, m is positive and n is non-negative, we can conclude that $k = 1$ and $m = n$.

$\square$

Modular Arithmetic

Halloween

- You have m sweets.
- There are k kids showing up for trick-or-treating.
- To keep everything fair, every kid gets the same amount of treats.
- You may enjoy the rest. :-)
- How much does every kid get, how much do you get?

Euclid's Division Lemma

Theorem (Euclid's division lemma)

For all integers a and b with $b \neq 0$
there are unique integers q and r
with $a = qb + r$ and $0 \leq r < |b|$.

Number a is called the dividend, b the divisor, q is the quotient and r the remainder.

Without proof.

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Examples:

- $a = 18, b = 5$
- $a = 5, b = 18$
- $a = -18, b = 5$
- $a = 18, b = -5$

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- Languages behave differently with negative operands!

Halloween

```
def share_sweets(no_kids, no_sweets):
    print("Each kid gets",
          no_sweets // no_kids,
          "of the sweets.")
    print("You may keep",
          no_sweets % no_kids,
          "of the sweets.")
```

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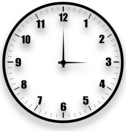
- Consider the clock:
  - It's now 3 o'clock



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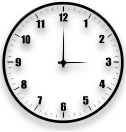

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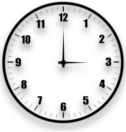

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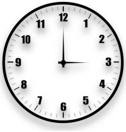
  - In the following, we will express this as $3 \equiv 15 \pmod{12}$



Congruence Modulo n – Definition

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For integer $n > 1$, two integers a and b
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Why is this the same concept as described in the clock example?!?

Congruence Corresponds to Equal Remainders

Theorem

For integers a and b and integer $n > 1$ it holds that $a \equiv b \pmod{n}$ iff there are $q, q', r \in \mathbb{Z}$ with

$$\begin{aligned} a &= qn + r \\ b &= q'n + r. \end{aligned}$$

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Proof sketch.

"$\Rightarrow$": If $n \mid a - b$ then there is a $k \in \mathbb{Z}$ with $kn = a - b$.

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Together, we get that $kn = qn + r - (q'n + r')$, which is the case iff $kn + r' = (q - q')n + r$. By Euclid's lemma, quotients and remainders are unique, so in particular $r' = r$.

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"$\Leftarrow$": If we subtract the equations, we get $a - b = (q - q')n$, so $n \mid a - b$ and $a \equiv b \pmod{n}$.

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Transitive: If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $n \mid a - b$ and $n \mid b - c$. Together, these imply that $n \mid a - b + b - c$. From $n \mid a - c$ we get $a \equiv c \pmod{n}$.

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For modulus n, the equivalence class of a is
$\bar{a}_n = \{\dots, a - 2n, a - n, a, a + n, a + 2n, \dots\}$.
Set $\bar{a}_n$ is called the congruence class or residue of a modulo n.

Compatibility with Operations

Theorem

Congruence modulo n is compatible with addition, subtraction, multiplication, translation, scaling and exponentiation, i. e. if $a \equiv b \pmod{n}$ and $a' \equiv b' \pmod{n}$ then

- $a + a' \equiv b + b' \pmod{n}$,
- $a - a' \equiv b - b' \pmod{n}$,
- $aa' \equiv bb' \pmod{n}$,
- $a + k \equiv b + k \pmod{n}$ *for all* $k \in \mathbb{Z}$,
- $ak \equiv bk \pmod{n}$ *for all* $k \in \mathbb{Z}$, *and*
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Congruence modulo n is a so-called congruence relation
($=$ equivalence relation compatible with operations).

Summary

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- m divides n (written $m \mid n$) if n is a multiple of m, i.e. there is an integer k with $n = mk$.
- Divisibility is compatible with multiplication and exponentiation.
- Divisibility over the natural numbers is a partial order.
- The modulo operation $a \bmod b$ corresponds to the remainder of Euclidean division.
- Congruence modulo n considers integers equivalent if they have with divisor n the same remainder.
- Congruence modulo n is an equivalence relation that is compatible with the arithmetic operations.