

Discrete Mathematics in Computer Science

B7. Operations on Relations

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Operations on Relations

Relations: Recap

- A relation over sets S_1, \dots, S_n is a set $R \subseteq S_1 \times \dots \times S_n$.

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- A **binary** relation is a relation over two sets.
- A **homogeneous** relation R over set S is a binary relation $R \subseteq S \times S$.

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With the standard relations for \mathbb{N}_0 , relation $=$ is the
complementary relation of \neq and $>$ the one of \leq .

Inverse of a Relation

Definition

Let $R \subseteq A \times B$ be a binary relation over A and B .

The **inverse relation** of R is the relation $R^{-1} \subseteq B \times A$ given by $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

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- The inverse of the $<$ relation over \mathbb{N}_0 is the $>$ relation.
- Relation R with xRy iff person x has a key for y .
Inverse: Q with aQb iff lock a can be opened by person b .

Composition of Relations

Definition (Composition of relations)

Let R_1 be a relation over A and B and R_2 a relation over B and C . The **composition of R_1 and R_2** is the relation $R_2 \circ R_1$ over A and C with:

$$R_2 \circ R_1 = \{(a, c) \mid \text{there is a } b \in B \text{ with} \\ (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

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How can we illustrate this graphically?

Composition of Relations: Example

$$S_1 = \{1, 2, 3, 4\}$$

$$S_2 = \{A, B, C, D, E\}$$

$$S_3 = \{a, b, c, d\}$$

$$R_1 = \{(1, A), (1, B), (3, B), (4, D)\} \text{ over } S_1 \text{ and } S_2$$

$$R_2 = \{(B, a), (C, c), (D, a), (D, d)\} \text{ over } S_2 \text{ and } S_3$$

$$R_2 \circ R_1 =$$

Composition is Associative

Theorem (Associativity of composition)

Let S_1, \dots, S_4 be sets and R_1, R_2, R_3 relations with $R_i \subseteq S_i \times S_{i+1}$.

Then

$$R_3 \circ (R_2 \circ R_1) = (R_3 \circ R_2) \circ R_1.$$

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Proof.

It holds that $(x_1, x_4) \in R_3 \circ (R_2 \circ R_1)$ iff there is an x_3 with $(x_1, x_3) \in R_2 \circ R_1$ and $(x_3, x_4) \in R_3$.



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As $(x_1, x_3) \in R_2 \circ R_1$ iff there is an x_2 with $(x_1, x_2) \in R_1$ and $(x_2, x_3) \in R_2$, we have overall that $(x_1, x_4) \in R_3 \circ (R_2 \circ R_1)$ iff there are x_2, x_3 with $(x_1, x_2) \in R_1$, $(x_2, x_3) \in R_2$ and $(x_3, x_4) \in R_3$.



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This is the case iff there is an x_2 with $(x_1, x_2) \in R_1$ and $(x_2, x_4) \in R_3 \circ R_2$, which holds iff $(x_1, x_4) \in (R_3 \circ R_2) \circ R_1$. □

Questions



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(Reflexive) Transitive Closure

Definition ((Reflexive) transitive closure)

Let R be a relation over set S .

The **transitive closure** R^+ of R is the **smallest relation over S that is transitive and has R as a subset.**

The **reflexive transitive closure** R^* of R is the **smallest relation over S that is reflexive, transitive and has R as a subset.**

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The (reflexive) transitive closure always exists. *Why?*

Example: If aRb specifies that there is a direct flight from a to b , what do R^+ and R^* express?

Transitive Closure and n -fold Composition

Define the n -fold composition of a relation R over S as

$$\begin{aligned} R^0 &= \{(x, x) \mid x \in S\} && \text{and} \\ R^i &= R \circ R^{i-1} && \text{for } i > 1. \end{aligned}$$

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$$R^i = R \circ R^{i-1} \quad \text{for } i > 1.$$

Theorem

Let R be a relation over set S .

Then $R^+ = \bigcup_{i=1}^{\infty} R^i$ and $R^* = \bigcup_{i=0}^{\infty} R^i$.

Without proof.

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- Highly relevant for **queries over relational databases**.
- For example, **join** operators combine relations based on common entries.
- Example for a **natural join**:

Employee

Name	Empld	DeptName
Harry	3415	Finance
Sally	2241	Sales
George	3401	Finance
Harriet	2202	Sales
Mary	1257	Human Resources

Dept

DeptName	Manager
Finance	George
Sales	Harriet
Production	Charles

Employee ⋈ Dept

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(Source: Wikipedia)

Summary

- Relations: general, binary, homogeneous
- Properties: reflexivity, symmetry, transitivity
(and related properties)
- Special relations: equivalence relations, order relations
- Operations: inverse, composition, transitive closure