Discrete Mathematics in Computer Science

B6. Equivalence and Order Relations

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Equivalence Relations

Relations: Recap

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  - transitive: if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$

Motivation

- Think of any attribute that two objects can have in common, e. g. their color.
- We could place the objects into distinct “buckets”, e. g. one bucket for each color.
- We also can define a relation $\sim$ such that $x \sim y$ iff x and y share the attribute, e. g.have the same color.
- Would this relation be
  - reflexive?
  - irreflexive?
  - symmetric?
  - asymmetric?
  - antisymmetric?
  - transitive?

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Examples:

- $\{(x, y) \mid x \text{ and } y \text{ have the same place of origin}\}$ over the set of all Swiss citizens
- $\{(x, y) \mid x \text{ and } y \text{ have the same parity}\}$ over $\mathbb{N}_0$
- $\{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), (2, 2), (2, 3), (3, 2), (3, 3)\}$ over $\{1, 2, \ldots, 5\}$

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- $\{(1,1),(1,4),(1,5),(4,1),(4,4),(4,5),(5,1),(5,4),(5,5),$ $(2,2),(2,3),(3,2),(3,3)\}$ over $\{1,2,\ldots,5\}$

Is this definition indeed what we want?
Does it allow us to partition the objects into buckets
(e. g. one "bucket" for all objects that share a specific color)?

Equivalence Classes

Definition (equivalence class)

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For any $x \in S$, the equivalence class of x is the set

$$[x]_\sim = \{y \in S \mid x \sim y\}.$$

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Consider

$\sim = \{(1,1),(1,4),(1,5),(4,1),(4,4),(4,5),(5,1),(5,4),(5,5),$
$(2,2),(2,3),(3,2),(3,3)\}$

over set $\{1,2,\dots,5\}$.

$[4]_\sim =$

Equivalence Classes: Properties

Let $\sim$ be an equivalence relation over set S and
$E = \{[x]_\sim \mid x \in S\}$ the set of all equivalence classes.

- Every element of S is in some equivalence class in E.
- Every element of S is in at most one equivalence class in E.
  $\rightsquigarrow$ homework assignment

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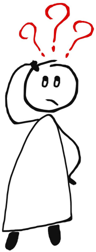
  $\leadsto$ homework assignment

$\Rightarrow$ Equivalence relations induce partitions
(not covered in this course).

Questions



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Order Relations

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- "Number x is not larger than number y."
  "Set S is a subset of set T."
  "Jerry runs at least as fast as Tom."
  "Pasta tastes better than Potatoes."

Partial Orders

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Which of these relations are partial orders?

- strict subset relation $\subset$ for sets
- not-less-than relation $\geq$ over $\mathbb{N}_0$
- $R = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$

Least and Greatest Element

Definition (Least and greatest element)

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- Why can we say the least element instead of a least element?

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Analogously: If there is a greatest element then is unique.

Minimal and Maximal Elements

Definition (Minimal/Maximal element of a set)

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A set can have several minimal elements and no least element.
Example?

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  - $\{1, 2\} \not\subseteq \{2, 3\}$ and $\{2, 3\} \not\subseteq \{1, 2\}$
- Relation $\leq$ is a total order, relation $\subseteq$ is not.

Total Order – Definition

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Definition (Total order)

A binary relation is a total order if it is total and a partial order.

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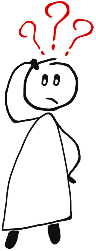

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Can a relation be both, a partial order and a strict order?

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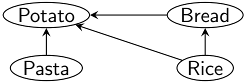

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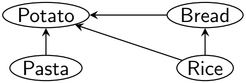

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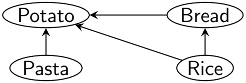

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- Example 2: $\subset$ relation for sets
- It doesn't work to simply require that the strict order is total. Why?

Strict Total Orders – Definition

Definition (Trichotomy)

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Definition (Strict total order)

A binary relation $\prec$ over S is a strict total order if $\prec$ is trichotomous and a strict order.

A strict total order completely ranks the elements of set S.

Example: $<$ relation over $\mathbb{N}_0$ gives the standard ordering $0, 1, 2, 3, \ldots$ of natural numbers.

Special Elements

Special elements are defined almost as for partial orders:

Definition (Least/greatest/minimal/maximal element of a set)

Let $\prec$ be a strict order over set S.

An element $x \in S$ is the least element of S
if for all $y \in S$ where $y \neq x$ it holds that $x \prec y$.

It is the greatest element of S if for all $y \in S$ where $y \neq x$, $y \prec x$.

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if there is no $y \in S$ with $y \prec x$.

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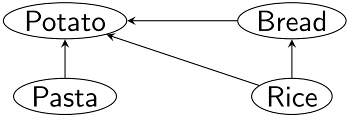
Special Elements – Example

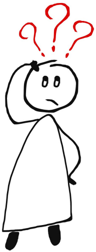
Consider again the previous example:

$S = \{\text{Pasta}, \text{Potato}, \text{Bread}, \text{Rice}\}$
$\prec\ = \{(\text{Pasta}, \text{Potato}), (\text{Bread}, \text{Potato}),$
$(\text{Rice}, \text{Potato}), (\text{Rice}, \text{Bread})\}$



Is there a least and a greatest element?
Which elements are maximal or minimal?

Questions



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- A partial order $x \preceq y$ is reflexive, antisymmetric and transitive.
  - If x is the greatest element of a set S, it is greater than every element: for all $y \in S$ it holds that $y \preceq x$.
  - If x is a maximal element of set S then it is not smaller than any other element y: there is no $y \in S$ with $x \preceq y$ and $x \neq y$.
  - A total order is a partial order without incomparable objects.

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  - A total order is a partial order without incomparable objects.
- A strict order is irreflexive, asymmetric and transitive.
  - Strict total orders and special elements are analogously defined as for partial orders but with a special treatment of equal elements.