Discrete Mathematics in Computer Science

B5. Relations

Malte Helmert, Gabriele Röger

University of Basel

October 16, 2023

B5.1 Relations

Relations: Informally

- Informally, a relation is some property that is true or false for an (ordered) collection of objects.
- We already know some relations, e. g.
  - $\subseteq$ relation for sets
  - $\leq$ relation for natural numbers
- These are examples of binary relations, considering pairs of objects.
- There are also relations of higher arity, e. g.
  - "$x + y = z$" for integers x, y, z.
  - "The name, address and office number belong to the same person."
- Relations are for example important for relational databases, semantic networks or knowledge representation and reasoning.

Relations

Definition (Relation)

Let $S_1, \dots, S_n$ be sets.

A relation over $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.

The arity of R is n.

- A relation of arity n is a set of n-tuples.
- The set contains the tuples
  for which the informal property is true.

Relations: Examples

- $\subseteq \; = \{(S, S') \mid S$ and S' are sets and for every $x \in S$ it holds that $x \in S'\}$
- $\leq \; = \{(x, y) \mid x, y \in \mathbb{N}_0$ and $x < y$ or $x = y\}$
- $R = \{(x, y, z) \mid x, y, z \in \mathbb{Z}$ and $x + y = z\}$
- $R' = \{$(Gabi Röger, Spiegelgasse 1, 04.005), (Malte Helmert, Spiegelgasse 1, 06.004), (Salomé Eriksson, Spiegelgasse 5, 04.003), (Claudia Grundke, Spiegelgasse 5, 04.001)$\}$

B5.2 Properties of Binary Relations

Binary Relation

A binary relation is a relation of arity 2:

Definition (binary relation)

A **binary relation** is a relation over two sets A and B.

- Instead of $(x, y) \in R$, we also write xRy, e. g.
  $x \leq y$ instead of $(x, y) \in \leq$
- If the sets are equal, we say "R is a binary relation over A" instead of "R is a binary relation over A and A".
- Such a relation over a set is also called a **homogeneous relation** or an **endorelation**.

Reflexivity

A reflexive relation relates every object to itself.

Definition (reflexive)

A binary relation R over set A is reflexive
if for all $a \in A$ it holds that $(a, a) \in R$.

Which of these relations are reflexive?

- $R = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$ over $\{a, b, c\}$
- $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- equality relation $=$ on natural numbers
- less-than relation $\leq$ on natural numbers
- strictly-less-than relation $<$ on natural numbers

Irreflexivity

A irreflexive relation never relates an object to itself.

Definition (irreflexive)

A binary relation R over set A is irreflexive
if for all $a \in A$ it holds that $(a, a) \notin R$.

Which of these relations are irreflexive?

- $R = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$ over $\{a, b, c\}$
- $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- equality relation $=$ on natural numbers
- less-than relation $\leq$ on natural numbers
- strictly-less-than relation $<$ on natural numbers

Symmetry

Definition (symmetric)

A binary relation R over set A is symmetric if for all $a, b \in A$ it holds that $(a, b) \in R$ iff $(b, a) \in R$.

Which of these relations are symmetric?

- $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- equality relation $=$ on natural numbers
- less-than relation $\leq$ on natural numbers
- strictly-less-than relation $<$ on natural numbers

Asymmetry and Antisymmetry

Definition (asymmetric and antisymmetric)

Let R be a binary relation over set A.

Relation R is asymmetric if for all $a, b \in A$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$.

Relation R is antisymmetric if for all $a, b \in A$ with $a \neq b$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$.

Which of these relations are asymmetric/antisymmetric?

- $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- equality relation $=$ on natural numbers
- less-than relation $\leq$ on natural numbers
- strictly-less-than relation $<$ on natural numbers

How do these properties relate to irreflexivity?

Transitivity

Definition

A binary relation R over set A is transitive
if it holds for all $a, b, c \in A$ that
if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Which of these relations are transitive?

- $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- equality relation $=$ on natural numbers
- less-than relation $\leq$ on natural numbers
- strictly-less-than relation $<$ on natural numbers

Summary

- A relation over sets $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.
- A binary relation is a relation over two sets.
- A binary relation over set S is a relation $R \subseteq S \times S$ and also called a homogeneous relation.
- A binary relation R over A is
  - reflexive if $(a, a) \in R$ for all $a \in A$,
  - irreflexive if $(a, a) \notin R$ for all $a \in A$,
  - symmetric if for all $a, b \in A$ it holds that $(a, b) \in R$ iff $(b, a) \in R$,
  - asymmetric if for all $a, b \in A$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$,
  - antisymmetric if for all $a, b \in A$ with $a \neq b$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$,
  - transitive if for all $a, b, c \in A$ it holds that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Special Classes of Relations

- Some important classes of relations are defined in terms of these properties.
  - Equivalence relation: reflexive, symmetric, transitive
  - Partial order: reflexive, antisymmetric, transitive
  - Strict order: irreflexive, asymmetric, transitive
  - ...
- We will consider these and other classes in detail.