Discrete Mathematics in Computer Science

B3. Cantor's Theorem

Malte Helmert, Gabriele Röger

University of Basel

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Cantor's Theorem

Countable Sets

We already know:

- Sets with the same cardinality as $\mathbb{N}_0$ are called countably infinite.
- A countable set is finite or countably infinite.
- Every subset of a countable set is countable.
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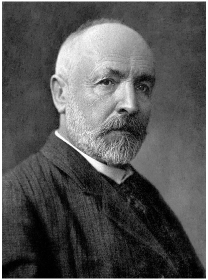
Open questions (to be resolved today):

- Do all infinite sets have the same cardinality?
- Does the power set of an infinite set S have the same cardinality as S?

Georg Cantor



- German mathematician (1845–1918)
- Proved that the rational numbers are countable.
- Proved that the real numbers are not countable.
- Cantor's Theorem: For every set S it holds that $|S| < |\mathcal{P}(S)|$.

Our Plan

- Understand Cantor's theorem
- Understand an important theoretical implication for computer science

Cantor's Diagonal Argument Illustrated on a Finite Set

$S = \{a, b, c\}$.

Consider an arbitrary function from S to $\mathcal{P}(S)$.
For example:

	a	b	c	
a	1	0	1	a mapped to $\{a, c\}$
b	1	1	0	b mapped to $\{a, b\}$
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We can identify an "unused" element of $\mathcal{P}(S)$.
Complement the entries on the main diagonal.

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We can identify an "unused" element of $\mathcal{P}(S)$.
Complement the entries on the main diagonal.

Works with every function from S to $\mathcal{P}(S)$.
$\rightarrow$ there cannot be a surjective function from S to $\mathcal{P}(S)$.
$\rightarrow$ there cannot be a bijection from S to $\mathcal{P}(S)$.

Cantor's Diagonal Argument on a Countably Infinite Set

$S = \mathbb{N}_0$.

Consider an arbitrary function from $\mathbb{N}_0$ to $\mathcal{P}(\mathbb{N}_0)$.
For example:

	0	1	2	3	4	...
0	1	0	1	0	1	...
1	1	1	0	1	0	...
2	0	1	0	1	0	...
3	1	1	0	0	0	...
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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⋮	⋮	⋮	⋮	⋮	⋮	⋱
	0	0	1	1	0	...

Complementing the entries on the main diagonal again results in an "unused" element of $\mathcal{P}(\mathbb{N}_0)$.

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For 1, consider function $f : S \to \mathcal{P}(S)$ with $f(x) = \{x\}$.
It maps distinct elements of S to distinct elements of $\mathcal{P}(S)$. ...

Cantor's Theorem

Proof (continued).

We show 2 by contradiction.
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Consider $M = \{x \mid x \in S, x \notin f(x)\}$ and note that $M \in \mathcal{P}(S)$.

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Proof (continued).

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Consider $M = \{x \mid x \in S, x \notin f(x)\}$ and note that $M \in \mathcal{P}(S)$.

Since f is bijective, it is surjective and there is an $x \in S$ with $f(x) = M$. Consider this x in a case distinction:

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If $x \in M$ then $x \notin f(x)$ by the definition of M. Since $f(x) = M$ this implies $x \notin M$. $\rightsquigarrow$ contradiction

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The assumption was false and we conclude that there is no bijection from S to $\mathcal{P}(S)$. □

Consequences of Cantor's Theorem

Infinite Sets can Have Different Cardinalities

There are infinitely many different cardinalities of infinite sets:

- $|\mathbb{N}_0| < |\mathcal{P}(\mathbb{N}_0)| < |\mathcal{P}(\mathcal{P}(\mathbb{N}_0))| < \ldots$
- $|\mathbb{N}_0| = \aleph_0 = \beth_0$
- $|\mathcal{P}(\mathbb{N}_0)| = \beth_1 (= |\mathbb{R}|)$
- $|\mathcal{P}(\mathcal{P}(\mathbb{N}_0))| = \beth_2$
- $\ldots$

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Why can we say so?

Decision Problems

"Intuitive Definition:" Decision Problem

A decision problem is a Yes-No question of the form
"Does the given input have a certain property?"

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- Input can be encoded as some finite string.
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- A computer program solves a decision problem if it terminates on every input and returns the correct answer.

More Problems than Programs I

- A computer program is given by a finite string.
- A decision problem corresponds to a set of strings.

More Problems than Programs II

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  - every subset of S corresponds to a separate decision problem
- By Cantor's theorem $|S| < |\mathcal{P}(S)|$,
  so there are more problems than programs.

Sets: Summary

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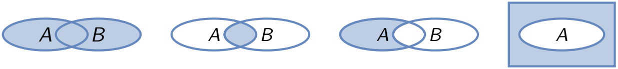
- Set operations: union, intersection, set difference, complement



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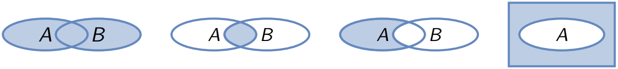

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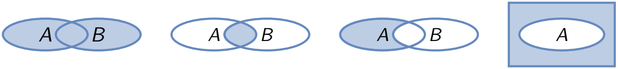

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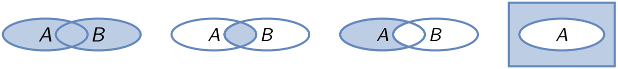

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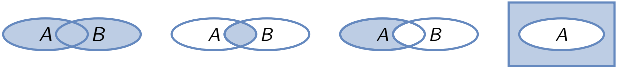

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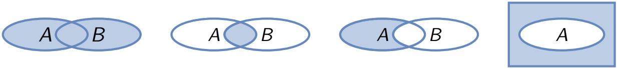

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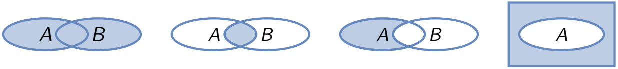

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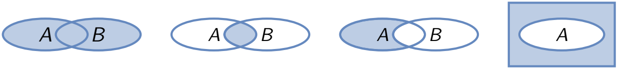

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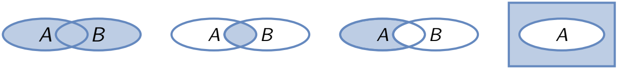

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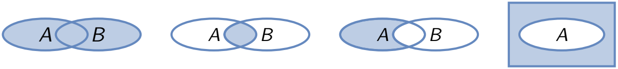

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