Discrete Mathematics in Computer Science

B2. Sets: Countability

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B2.1 Comparing Cardinality

Finite Sets Revisited

We already know:

- The cardinality $|S|$ measures the size of set S.
- A set is finite if it has a finite number of elements.
- The cardinality of a finite set
  is the number of elements it contains.

A set is infinite if it has an infinite number of elements.

Do all infinite sets have the same cardinality?

Comparing the Cardinality of Sets

- Consider $A = \{1, 2\}$ and $B = \{\text{dog}, \text{cat}, \text{mouse}\}$.
- We can map distinct elements of A to distinct elements of B:

$$1 \mapsto \text{dog}$$
$$2 \mapsto \text{cat}$$

- We call this an injective function from A to B:
  - every element of A is mapped to an element of B;
  - different elements of A are mapped to different elements of B.

Comparing Cardinality

Definition (cardinality not larger)

Set A has cardinality less than or equal to the cardinality of set B ($|A| \leq |B|$), if there is an injective function from A to B.

Comparing the Cardinality of Sets

- $A = \{1, 2, 3\}$ and $B = \{\text{dog}, \text{cat}, \text{mouse}\}$ have cardinality 3.
- We can pair their elements:

x		$f(x)$
1	$\leftrightarrow$	dog
2	$\leftrightarrow$	cat
3	$\leftrightarrow$	mouse

- We call such a mapping a bijection from A to B.
  - Each element of A is paired with exactly one element of set B.
  - Each element of B is paired with exactly one element of A.
  - Mathematically:
    - f is a function from A to B.
    - f is injective: if $x \neq y$ then $f(x) \neq f(y)$.
    - f is surjective: for all $y \in B$ there is an $x \in A$ with $f(x) = y$.
- If there is a bijection from A to B there is one from B to A.

Equinumerous Sets

We use the existence of a bijection also as criterion for infinite sets:

Definition (equinumerous sets)

Two sets A and B have the same cardinality ($|A| = |B|$) if there exists a bijection from A to B.

Such sets are called equinumerous.

Definition (strictly smaller cardinality)

Set A has cardinality strictly less than the cardinality of set B ($|A| < |B|$), if $|A| \leq |B|$ and $|A| \neq |B|$.

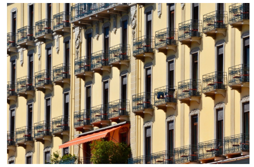
Consider set A and object $e \notin A$. Is $|A| < |A \cup \{e\}|$?

B2.2 Hilbert's Hotel

Hilbert's Hotel

Our intuition for finite sets does not always work for infinite sets.

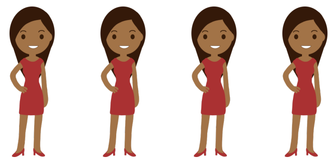
- If in a hotel all rooms are occupied then it cannot accomodate additional guests.
- But Hilbert's Grand Hotel has infinitely many rooms.
- All these rooms are occupied.



One More Guest Arrives

- Every guest moves from her current room n to room $n + 1$.
- Room 1 is then free.
- The new guest gets room 1.

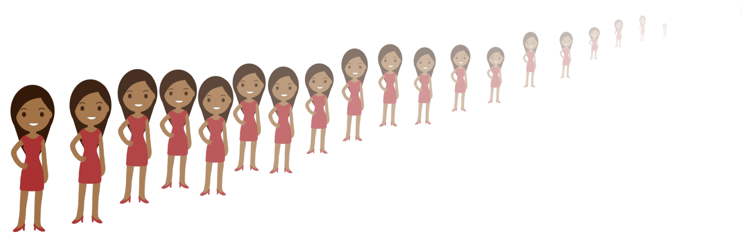
Four More Guests Arrive



- Every guest moves from her current room n to room $n + 4$.
- Rooms 1 to 4 are no longer occupied and can be used for the new guests.

$\rightarrow$ Works for any finite number of additional guests.

An Infinite Number of Guests Arrives



- Every guest moves from her current room n to room $2n$.
- The infinitely many rooms with odd numbers are now available.
- The new guests fit into these rooms.

Can we Go further?

What if . . .

- infinitely many coaches, each with an infinite number of guests
- infinitely many ferries, each with an infinite number of coaches, each with infinitely many guests
- . . .

. . . arrive?

There are strategies for all these situations
as long as with "infinite" we mean "countably infinite"
and there is a finite number of layers.

B2.3 Countable Sets

Comparing Cardinality

- Two sets A and B have the same cardinality if their elements can be paired (i.e. there is a bijection from A to B).
- Set A has a strictly smaller cardinality than set B if
  - we can map distinct elements of A to distinct elements of B (i.e. there is an injective function from A to B), and
  - $|A| \neq |B|$.

- This clearly makes sense for finite sets.
- What about infinite sets? Do they even have different cardinalities?

Countable and Countably Infinite Sets

Definition (countably infinite and countable)

A set A is countably infinite if $|A| = |\mathbb{N}_0|$.

A set A is countable if $|A| \leq |\mathbb{N}_0|$.

A set is countable if it is finite or countably infinite.

- We can count the elements of a countable set one at a time.
- The objects are "discrete" (in contrast to "continuous").
- Discrete mathematics deals with all kinds of countable sets.

Set of Even Numbers

- $\mathit{even} = \{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}$
- Obviously: $\mathit{even} \subset \mathbb{N}_0$
- Intuitively, there are twice as many natural numbers as even numbers — no?
- Is $|\mathit{even}| < |\mathbb{N}_0|$?

Set of Even Numbers

Theorem (set of even numbers is countably infinite)

The set of all even natural numbers is countably infinite, i. e. $|\{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}| = |\mathbb{N}_0|$.

Proof Sketch.

We can pair every natural number n with the even number $2n$. □

Set of Perfect Squares

Theorem (set of perfect squares is countably infininite)

The set of all perfect squares is countably infinite, i. e. $|\{n^2 \mid n \in \mathbb{N}_0\}| = |\mathbb{N}_0|$.

Proof Sketch.

We can pair every natural number n with square number n^2. □

Subsets of Countable Sets are Countable

In general:

Theorem (subsets of countable sets are countable)

Let A be a countable set. Every set B with $B \subseteq A$ is countable.

Proof.

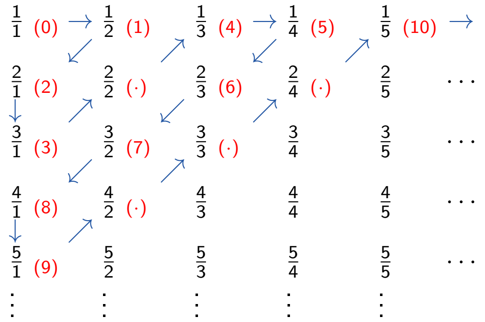
Since A is countable there is an injective function f from A to $\mathbb{N}_0$.
The restriction of f to B is an injective function from B to $\mathbb{N}_0$. □

Set of the Positive Rationals

Theorem (set of positive rationals is countably infininite)

Set $\mathbb{Q}_+ = \{n \mid n \in \mathbb{Q} \text{ and } n > 0\} = \{p/q \mid p, q \in \mathbb{N}_1\}$ is countably infinite.

Proof idea.



□

Union of Two Countable Sets is Countable

Theorem (union of two countable sets countable)

Let A and B be countable sets. Then $A \cup B$ is countable.

Proof sketch.

As A and B are countable there is an injective function f_A from A to $\mathbb{N}_0$, analogously f_B from B to $\mathbb{N}_0$.

We define function $f_{A\cup B}$ from $A \cup B$ to $\mathbb{N}_0$ as

$$f_{A\cup B}(e) = \begin{cases} 2f_A(e) & \text{if } e \in A \\ 2f_B(e) + 1 & \text{otherwise} \end{cases}$$

This $f_{A\cup B}$ is an injective function from $A \cup B$ to $\mathbb{N}_0$. □

Integers and Rationals

Theorem (sets of integers and rationals are countably infinite)

The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.

Without proof ($\rightsquigarrow$ exercises)

Union of More than Two Sets

Definition (arbitrary unions)

Let M be a set of sets. The union $\bigcup_{S \in M} S$ is the set with

$$x \in \bigcup_{S \in M} S \text{ iff exists } S \in M \text{ with } x \in S.$$

Countable Union of Countable Sets

Theorem

Let M be a countable set of countable sets.
Then $\bigcup_{S \in M} S$ is countable.

We proof this formally after we have studied functions.

Set of all Binary Trees is Countable

Theorem (set of all binary trees is countable)

The set $B = \{b \mid b \text{ is a binary tree}\}$ is countable.

Proof.

For $n \in \mathbb{N}_0$ the set B_n of all binary trees with n leaves is finite.

With $M = \{B_i \mid i \in \mathbb{N}_0\}$ the set of all binary trees is $B = \bigcup_{B' \in M} B'$.

Since M is a countable set of countable sets, B is countable. □

And Now?

We have seen several countably infinite sets.

What about our original questions?

- Do all infinite sets have the same cardinality?
- Are they all countably infinite?

Summary

- Set A has cardinality less than or equal the cardinality of set B ($|A| \leq |B|$), if there is an injective function from A to B.
- Sets A and B have the same cardinality ($|A| = |B|$) if there exists a bijection from A to B.
- Our intuition for finite sets does not always work for infinite sets.
- A set is countable if it has at most cardinality $|\mathbb{N}_0|$.
- If a set is countable and infinite, it is countably infinite.
- Set $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.
- Every subset of a countable set is countable.
- Every countable union of countable sets is countable, in particular, the union of two countable sets is countable.