Discrete Mathematics in Computer Science

B1. Sets: Foundations

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Sets

Important Building Blocks of Discrete Mathematics

- sets
- relations
- functions

Sets

Definition

A set is an unordered collection of distinct objects.

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- unorderd: no notion of a "first" or "second" object, e. g. $\{Alice, Bob, Charly\} = \{Charly, Bob, Alice\}$
- distinct: each object contained at most once, e. g. $\{Alice, Bob, Charly\} = \{Alice, Charly, Bob, Alice\}$

Notation

- Specification of sets
  - explicit, listing all elements, e. g. $A = \{1, 2, 3\}$
  - implicit with set-builder notation, specifying a property characterizing all elements, e. g. $A = \{x \mid x \in \mathbb{N}_0 \text{ and } 1 \leq x \leq 3\}$, $B = \{n^2 \mid n \in \mathbb{N}_0\}$
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Question: Is it true that $1 \in \{\{1, 2\}, 3\}$?

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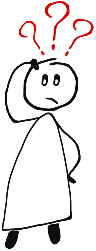
- Rational numbers $\mathbb{Q} = \{n/d \mid n \in \mathbb{Z}, d \in \mathbb{N}_1\}$
- Real numbers $\mathbb{R} = (-\infty, \infty)$
  Why do we use interval notation?
  Why didn't we introduce it before?

Questions



Questions?

Russell’s Paradox

Excursus: Barber Paradox

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In a town there is only one barber, who is male.

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We can exploit the self-reference to derive a contradiction.

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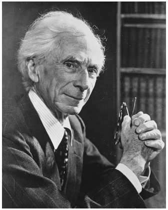

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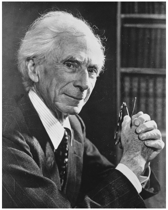

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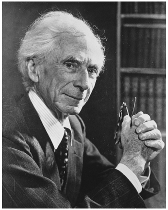

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Assume that S is a set.
If $S \notin S$ then $S \in S \rightsquigarrow$ Contradiction
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Hence, there is no such set S.

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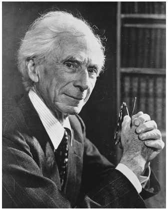

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Hence, there is no such set S.

$\rightarrow$ Not every property used in set-builder notation defines a set.

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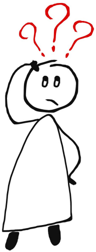

Questions?

Relations on Sets

Equality

Definition (Axiom of Extensionality)

Two sets A and B are equal (written $A = B$)
if every element of A is an element of B and vice versa.

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We write $A \neq B$ to indicate that A and B are not equal.

Subsets and Supersets

- $A \subseteq B$: A is a subset of B,
  i. e., every element of A is an element of B
- $A \subset B$: A is a strict subset of B,
  i. e., $A \subseteq B$ and $A \neq B$.
- $A \supseteq B$: A is a superset of B if $B \subseteq A$.
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We write $A \nsubseteq B$ to indicate that A is not a subset of B.

Analogously: $\not\subset$, $\nsupseteq$, $\not\supset$

Power Set

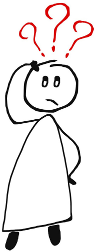
Definition (Power Set)

The power set $\mathcal{P}(S)$ of a set S is the set of all subsets of S. That is,

$$\mathcal{P}(S) = \{M \mid M \subseteq S\}.$$

Example: $\mathcal{P}(\{a, b\}) =$

Questions



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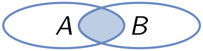
- intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



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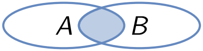

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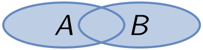

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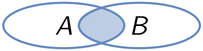

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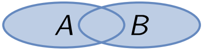
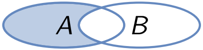


- set difference $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



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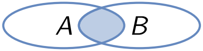

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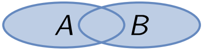

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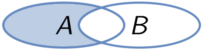
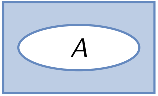


- complement $\overline{A} = B \setminus A$, where $A \subseteq B$ and B is the set of all considered objects (in a given context)



Properties of Set Operations: Commutativity

Theorem (Commutativity of $\cup$ and $\cap$)

For all sets A and B it holds that

- $A \cup B = B \cup A$ *and*
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Question: Is the set difference also commutative,
i. e. is $A \setminus B = B \setminus A$ for all sets A and B?

Properties of Set Operations: Associativity

Theorem (Associativity of $\cup$ and $\cap$)

For all sets A, B and C it holds that

- $(A \cup B) \cup C = A \cup (B \cup C)$ *and*
- $(A \cap B) \cap C = A \cap (B \cap C)$.

Properties of Set Operations: Distributivity

Theorem (Union distributes over intersection and vice versa)

For all sets A, B *and* C *it holds that*

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *and*
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Properties of Set Operations: De Morgan's Law

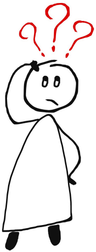
Augustus De Morgan
British mathematician (1806-1871)

Theorem (De Morgan's Law)

For all sets A and B it holds that

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$ *and*
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Questions



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Finite Sets

Cardinality of Sets

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The cardinality of a finite set is the number of elements it contains.

- $|\emptyset| =$
- $|\{x \mid x \in \mathbb{N}_0 \text{ and } 2 \leq x < 5\}| =$
- $|\{3, 0, \{1, 3\}\}| =$
- $|\mathcal{P}(\{1, 2\})| =$

Cardinality of the Union of Sets

Theorem

For finite sets A and B it holds that $|A \cup B| = |A| + |B| - |A \cap B|$.

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Corollary

If finite sets A and B are disjoint then $|A \cup B| = |A| + |B|$.

Cardinality of the Power Set

Theorem

Let S be a finite set. Then $|\mathcal{P}(S)| = 2^{|S|}$.

Proof sketch.

We can construct a subset S' by iterating over all elements e of S and deciding whether e becomes a member of S' or not.

We make $|S|$ independent decisions, each between two options. Hence, there are $2^{|S|}$ possible outcomes.

Every subset of S can be constructed this way and different choices lead to different sets. Thus, $|\mathcal{P}(S)| = 2^{|S|}$. □

Alternative Proof by Induction

Proof.

By induction over $|S|$.

Basis ($|S| = 0$): Then $S = \emptyset$ and $|\mathcal{P}(S)| = |\{\emptyset\}| = 1 = 2^0$.

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Inductive Step ($n \rightarrow n+1$):

Let S' be an arbitrary set with $|S'| = n + 1$ and
let e be an arbitrary member of S'.

Let further $S = S' \setminus \{e\}$ and $X = \{S'' \cup \{e\} \mid S'' \in \mathcal{P}(S)\}$.

Then $\mathcal{P}(S') = \mathcal{P}(S) \cup X$. As $\mathcal{P}(S)$ and X are disjoint and $|X| = |\mathcal{P}(S)|$, it holds that $|\mathcal{P}(S')| = 2|\mathcal{P}(S)|$.

Since $|S| = n$, we can use the IH and get

$$|\mathcal{P}(S')| = 2 \cdot 2^{|S|} = 2 \cdot 2^n = 2^{n+1} = 2^{|S'|}.$$

$\square$

Enumerating all Subsets

Determine a one-to-one mapping between numbers $0, \dots, 2^{|S|} - 1$ and all subsets of finite set S:

- Consider the binary representation of numbers $0, \dots, 2^{|S|} - 1$.
- Associate every bit with a different element of S.
- Every number is mapped to the set that contains exactly the elements associated with the 1-bits.

$S = \{a, b, c\}$

decimal	binary abc	set
0	000	$\{\}$
1	001	$\{c\}$
2	010	$\{b\}$
3	011	$\{b, c\}$
4	100	$\{a\}$
5	101	$\{a, c\}$
6	110	$\{a, b\}$
7	111	$\{a, b, c\}$

Computer Representation as Bit String

Same representation as in enumeration of all subsets:

- Required: Fixed universe U of possible elements
- Represent sets as bitstrings of length $|U|$
- Associate every bit with one object from the universe
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- $U = \{o_0, \ldots, o_9\}$
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How can the set operations be implemented?

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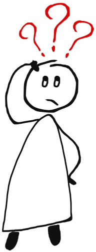

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Summary

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- Sets are unordered collections of distinct objects.
- Important set relations: equality ($=$), subset ($\subseteq$), superset ($\supseteq$) and strict variants ($\subset$ and $\supset$)
- The power set of a set S is the set of all subsets of S.
- Important set operations are intersection, union, set difference and complement.
    - Union and intersection are commutative and associative.
    - Union distributes over intersection and vice versa.
    - De Morgan's law for complement of union or intersection.
- The number of elements in a finite set is called its cardinality.
- Sets over a finite universe can be represented as bit strings.
  $\rightarrow$ also useful for enumerating all subsets