Discrete Mathematics in Computer Science

B1. Sets: Foundations

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October 2, 2023

B1.1 Sets

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B1.1 Sets

Important Building Blocks of Discrete Mathematics

- sets
- relations
- functions

Sets

Definition

A set is an unordered collection of distinct objects.

- unorderd: no notion of a “first” or “second” object,
  e. g. $\{Alice, Bob, Charly\} = \{Charly, Bob, Alice\}$
- distinct: each object contained at most once,
  e. g. $\{Alice, Bob, Charly\} = \{Alice, Charly, Bob, Alice\}$

Notation

- Specification of sets
  - explicit, listing all elements, e. g. $A = \{1, 2, 3\}$
  - implicit with set-builder notation,
    specifying a property characterizing all elements,
    e. g. $A = \{x \mid x \in \mathbb{N}_0 \text{ and } 1 \leq x \leq 3\}$,
    $B = \{n^2 \mid n \in \mathbb{N}_0\}$
  - implicit, as a sequence with dots,
    e. g. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
  - implicit with an inductive definition
- $e \in M$: e is in set M (an element of the set)
- $e \notin M$: e is not in set M
- empty set $\emptyset = \{\}$

Question: Is it true that $1 \in \{\{1, 2\}, 3\}$?

Special Sets

- Natural numbers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$
- Integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- Positive integers $\mathbb{Z}_+ = \mathbb{N}_1 = \{1, 2, \dots\}$
- Rational numbers $\mathbb{Q} = \{n/d \mid n \in \mathbb{Z}, d \in \mathbb{N}_1\}$
- Real numbers $\mathbb{R} = (-\infty, \infty)$
  Why do we use interval notation?
  Why didn't we introduce it before?

B1.2 Russell's Paradox

Excursus: Barber Paradox

Barber Paradox

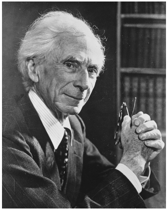
In a town there is only one barber, who is male.

The barber shaves all men in the town,
and only those, who do not shave themselves.

Who shaves the barber?

We can exploit the self-reference to derive a contradiction.

Russell's Paradox



Bertrand Russell

Question

Is the collection of all sets that do not contain themselves as a member a set?

Is $S = \{M \mid M \text{ is a set and } M \notin M\}$ a set?

Assume that S is a set.
If $S \notin S$ then $S \in S \rightsquigarrow$ Contradiction
If $S \in S$ then $S \notin S \rightsquigarrow$ Contradiction
Hence, there is no such set S.

$\rightarrow$ Not every property used in set-builder notation defines a set.

B1.3 Relations on Sets

Equality

Definition (Axiom of Extensionality)

Two sets A and B are equal (written $A = B$)
if every element of A is an element of B and vice versa.

Two sets are equal if they contain the same elements.

We write $A \neq B$ to indicate that A and B are not equal.

Subsets and Supersets

- $A \subseteq B$: A is a subset of B,
  i. e., every element of A is an element of B
- $A \subset B$: A is a strict subset of B,
  i. e., $A \subseteq B$ and $A \neq B$.
- $A \supseteq B$: A is a superset of B if $B \subseteq A$.
- $A \supset B$: A is a strict superset of B if $B \subset A$.

We write $A \not\subseteq B$ to indicate that A is not a subset of B.

Analogously: $\not\subset$, $\not\supseteq$, $\not\supset$

Power Set

Definition (Power Set)

The power set $\mathcal{P}(S)$ of a set S is the set of all subsets of S.
That is,

$$\mathcal{P}(S) = \{M \mid M \subseteq S\}.$$

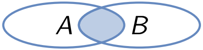
Example: $\mathcal{P}(\{a, b\}) =$

B1.4 Set Operations

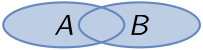
Set Operations

Set operations allow us to express sets in terms of other sets

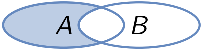
- intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

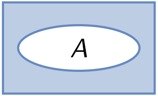
  If $A \cap B = \emptyset$ then A and B are disjoint.
- union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

- set difference $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

- complement $\overline{A} = B \setminus A$, where $A \subseteq B$ and
  B is the set of all considered objects (in a given context)

Properties of Set Operations: Commutativity

Theorem (Commutativity of $\cup$ and $\cap$)

For all sets A and B it holds that

- $A \cup B = B \cup A$ *and*
- $A \cap B = B \cap A$.

Question: Is the set difference also commutative,
i. e. is $A \setminus B = B \setminus A$ for all sets A and B?

Properties of Set Operations: Associativity

Theorem (Associativity of $\cup$ and $\cap$)

For all sets A, B and C it holds that

- $(A \cup B) \cup C = A \cup (B \cup C)$ *and*
- $(A \cap B) \cap C = A \cap (B \cap C)$.

Properties of Set Operations: Distributivity

Theorem (Union distributes over intersection and vice versa)

For all sets A, B and C it holds that

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *and*
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Properties of Set Operations: De Morgan's Law

Augustus De Morgan
British mathematician (1806-1871)

Theorem (De Morgan's Law)

For all sets A and B it holds that

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$ *and*
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

B1.5 Finite Sets

Cardinality of Sets

The cardinality $|S|$ measures the size of set S.

A set is finite if it has a finite number of elements.

Definition (Cardinality)

The cardinality of a finite set is the number of elements it contains.

- $|\emptyset| =$
- $|\{x \mid x \in \mathbb{N}_0 \text{ and } 2 \leq x < 5\}| =$
- $|\{3, 0, \{1, 3\}\}| =$
- $|\mathcal{P}(\{1, 2\})| =$

Cardinality of the Union of Sets

Theorem

For finite sets A and B it holds that $|A \cup B| = |A| + |B| - |A \cap B|$.

Corollary

If finite sets A and B are disjoint then $|A \cup B| = |A| + |B|$.

Cardinality of the Power Set

Theorem

Let S be a finite set. Then $|\mathcal{P}(S)| = 2^{|S|}$.

Proof sketch.

We can construct a subset S' by iterating over all elements e of S and deciding whether e becomes a member of S' or not.

We make $|S|$ independent decisions, each between two options. Hence, there are $2^{|S|}$ possible outcomes.

Every subset of S can be constructed this way and different choices lead to different sets. Thus, $|\mathcal{P}(S)| = 2^{|S|}$. $\square$

Alternative Proof by Induction

Proof.

By induction over $|S|$.

Basis ($|S| = 0$): Then $S = \emptyset$ and $|\mathcal{P}(S)| = |\{\emptyset\}| = 1 = 2^0$.

IH: For all sets S with $|S| = n$, it holds that $|\mathcal{P}(S)| = 2^{|S|}$.

Inductive Step ($n \to n+1$):

Let S' be an arbitrary set with $|S'| = n + 1$ and
let e be an arbitrary member of S'.

Let further $S = S' \setminus \{e\}$ and $X = \{S'' \cup \{e\} \mid S'' \in \mathcal{P}(S)\}$.

Then $\mathcal{P}(S') = \mathcal{P}(S) \cup X$. As $\mathcal{P}(S)$ and X are disjoint and $|X| = |\mathcal{P}(S)|$, it holds that $|\mathcal{P}(S')| = 2|\mathcal{P}(S)|$.

Since $|S| = n$, we can use the IH and get

$$|\mathcal{P}(S')| = 2 \cdot 2^{|S|} = 2 \cdot 2^n = 2^{n+1} = 2^{|S'|}.$$

$\square$

Enumerating all Subsets

Determine a one-to-one mapping between numbers $0, \dots, 2^{|S|} - 1$ and all subsets of finite set S:

- Consider the binary representation of numbers $0, \dots, 2^{|S|} - 1$.
- Associate every bit with a different element of S.
- Every number is mapped to the set that contains exactly the elements associated with the 1-bits.

$S = \{a, b, c\}$

decimal	binary abc	set
0	000	$\{\}$
1	001	$\{c\}$
2	010	$\{b\}$
3	011	$\{b, c\}$
4	100	$\{a\}$
5	101	$\{a, c\}$
6	110	$\{a, b\}$
7	111	$\{a, b, c\}$

Computer Representation as Bit String

Same representation as in enumeration of all subsets:

- Required: Fixed universe U of possible elements
- Represent sets as bitstrings of length $|U|$
- Associate every bit with one object from the universe
- Each bit is 1 iff the corresponding object is in the set

Example:

- $U = \{o_0, \dots, o_9\}$
- Associate the i-th bit (0-indexed, from left to right) with o_i
- $\{o_2, o_4, o_5, o_9\}$ is represented as:
  0010110001

How can the set operations be implemented?

Summary

- Sets are unordered collections of distinct objects.
- Important set relations: equality ($=$), subset ($\subseteq$), superset ($\supseteq$) and strict variants ($\subset$ and $\supset$)
- The power set of a set S is the set of all subsets of S.
- Important set operations are intersection, union, set difference and complement.
  - Union and intersection are commutative and associative.
  - Union distributes over intersection and vice versa.
  - De Morgan's law for complement of union or intersection.
- The number of elements in a finite set is called its cardinality.
- Sets over a finite universe can be represented as bit strings.
  $\rightarrow$ also useful for enumerating all subsets