# Planning and Optimization

## C7. Symbolic Search: Full Algorithm

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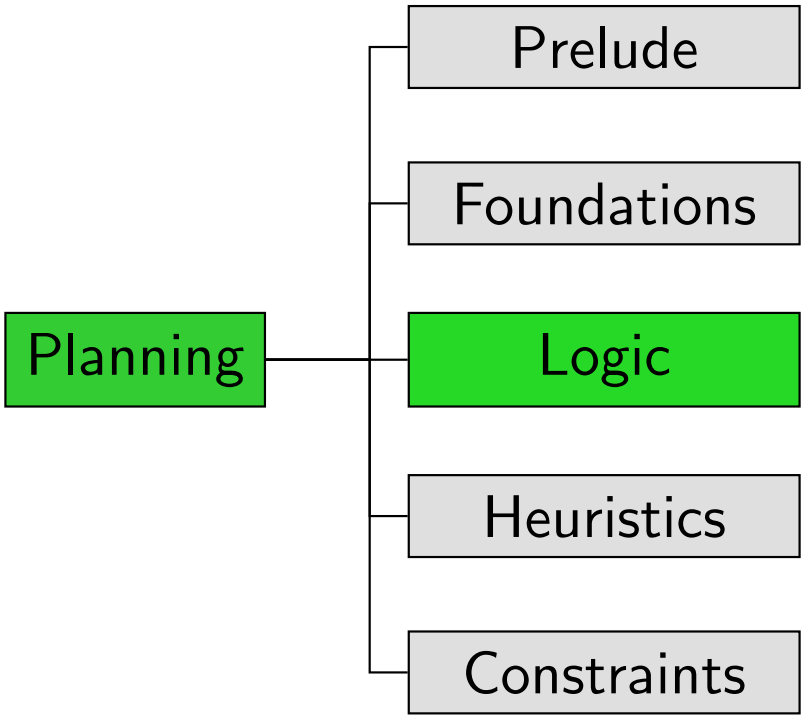
# Content of this Course



# Devising a Symbolic Search Algorithm

- We now put the pieces together to build a symbolic search algorithm for propositional planning tasks.
- use BDDs as a black box data structure:
  - care about provided operations and their time complexity
  - do not care about their internal implementation
- Efficient implementations are available as libraries, e.g.:
  - CUDD, a high-performance BDD library
  - libbdd, shipped with Ubuntu Linux

# Basic BDD Operations

# BDD Operations: Preliminaries

- All BDDs work on a fixed and totally ordered set of propositional variables.
- Complexity of operations given in terms of:
  - $k$, the number of BDD variables
  - $\|B\|$, the number of nodes in the BDD $B$

# BDD Operations (1)

BDD operations: logical/set atoms

- bdd-true(): build BDD representing all assignments
  - in logic: $\top$
  - time complexity: $O(1)$
- bdd-false(): build BDD representing $\emptyset$
  - in logic: $\bot$
  - time complexity: $O(1)$
- bdd-atom($v$): build BDD representing $\{s \mid s(v) = 1\}$
  - in logic: $v$
  - time complexity: $O(1)$

# BDD Operations (2)

BDD operations: logical/set connectives

- bdd-complement($B$): build BDD representing $\overline{r(B)}$
  - in logic: $\neg\varphi$
  - time complexity: $O(\|B\|)$
- bdd-union($B$, $B'$): build BDD representing $r(B) \cup r(B')$
  - in logic: $(\varphi \vee \psi)$
  - time complexity: $O(\|B\| \cdot \|B'\|)$
- analogously:
  - bdd-intersection($B$, $B'$): $r(B) \cap r(B')$, $(\varphi \wedge \psi)$
  - bdd-setdifference($B$, $B'$): $r(B) \setminus r(B')$, $(\varphi \wedge \neg\psi)$
  - bdd-implies($B$, $B'$): $\overline{r(B)} \cup r(B')$, $(\varphi \rightarrow \psi)$
  - bdd-equiv($B$, $B'$): $(r(B) \cap r(B')) \cup (\overline{r(B)} \cap \overline{r(B')})$, $(\varphi \leftrightarrow \psi)$

# BDD Operations (3)

BDD operations: Boolean tests

- bdd-includes($B$, $I$): return **true** iff $I \in r(B)$
  - in logic: $I \models \varphi$?
  - time complexity: $O(k)$
- bdd-equals($B$, $B'$): return **true** iff $r(B) = r(B')$
  - in logic: $\varphi \equiv \psi$?
  - time complexity: $O(1)$ (due to canonical representation)

# Conditioning: Formulas

The last two basic BDD operations are a bit more unusual and require some preliminary remarks.

Conditioning a variable $v$ in a formula $\varphi$ to $\mathbf{T}$ or $\mathbf{F}$, written $\varphi[\mathbf{T}/v]$ or $\varphi[\mathbf{F}/v]$, means restricting $v$ to a particular truth value:

Examples:

- $(A \wedge (B \vee \neg C))[\mathbf{T}/B] = (A \wedge (\top \vee \neg C)) \equiv A$
- $(A \wedge (B \vee \neg C))[\mathbf{F}/B] = (A \wedge (\bot \vee \neg C)) \equiv A \wedge \neg C$

# Conditioning: Sets of Assignments

We can define the same operation for sets of assignments $S$: $S[\mathbf{F}/v]$ and $S[\mathbf{T}/v]$ restrict $S$ to elements with the given value for $v$ and remove $v$ from the domain of definition:

Example:

- $S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}, C \mapsto \mathbf{F}\},$
  $\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{F}\},$
  $\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$

$\rightsquigarrow$ $S[\mathbf{T}/B] = \{\{A \mapsto \mathbf{T}, C \mapsto \mathbf{F}\},$
$\{A \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$

# Forgetting

Forgetting (a.k.a. existential abstraction) is similar to conditioning: we allow either truth value for $v$ and remove the variable.

We write this as $\exists v\, \varphi$ (for formulas) and $\exists v\, S$ (for sets).

Formally:

- $\exists v\, \varphi = \varphi[\mathbf{T}/v] \vee \varphi[\mathbf{F}/v]$
- $\exists v\, S = S[\mathbf{T}/v] \cup S[\mathbf{F}/v]$

# Forgetting: Example

Examples:

- $S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}, C \mapsto \mathbf{F}\},$
  $\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{F}\},$
  $\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$

$\rightsquigarrow$ $\exists B\, S = \{\{A \mapsto \mathbf{F}, C \mapsto \mathbf{F}\},$
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$\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}\}\}$

# BDD Operations (4)

BDD operations: conditioning and forgetting

- bdd-condition($B$, $v$, $t$) where $t \in \{\mathbf{T}, \mathbf{F}\}$:
  build BDD representing $r(B)[t/v]$
  - in logic: $\varphi[t/v]$
  - time complexity: $O(\|B\|)$
- bdd-forget($B$, $v$):
  build BDD representing $\exists v\, r(B)$
  - in logic: $\exists v\, \varphi \quad (= \varphi[\mathbf{T}/v] \vee \varphi[\mathbf{F}/v])$
  - time complexity: $O(\|B\|^2)$

# Formulas and Singletons

# Formulas to BDDs

- With the logical/set operations, we can convert propositional formulas $\varphi$ into BDDs representing the models of $\varphi$.
- We denote this computation with bdd-formula($\varphi$).
- Each individual logical connective takes polynomial time, but converting a full formula of length $n$ can take $O(2^n)$ time. (How is this possible?)

# Singleton BDDs

- We can convert a single truth assignment $I$ into a BDD representing $\{I\}$ by computing the conjunction of all literals true in $I$ (using bdd-atom, bdd-complement and bdd-intersection).
- We denote this computation with bdd-singleton($I$).
- When done in the correct order, this takes time $O(k)$.

# Renaming

# Renaming

We will need to support one final operation on formulas: renaming.

Renaming $X$ to $Y$ in formula $\varphi$, written $\varphi[X \to Y]$, means replacing all occurrences of $X$ by $Y$ in $\varphi$.

We require that $Y$ is not present in $\varphi$ initially.

Example:

- $\varphi = (A \wedge (B \vee \neg C))$

$\rightsquigarrow$ $\varphi[A \to D] = (D \wedge (B \vee \neg C))$

# How Hard Can That Be?

- For formulas, renaming is a simple (linear-time) operation.
- For a BDD $B$, it is equally simple ($O(\|B\|)$) when renaming between variables that are adjacent in the variable order.
- In general, it requires $O(\|B\|^2)$, using the equivalence $\varphi[X \to Y] \equiv \exists X(\varphi \wedge (X \leftrightarrow Y))$

# Symbolic Breadth-first Search

# Planning Task State Variables vs. BDD Variables

Consider propositional planning task $\langle V, I, O, \gamma \rangle$ with states $S$.

In symbolic planning, we have two BDD variables $v$ and $v'$ for every state variable $v \in V$ of the planning task.

- use unprimed variables $v$ to describe sets of states: $\{s \in S \mid \text{some property}\}$
- use combinations of unprimed and primed variables $v$, $v'$ to describe sets of state pairs: $\{\langle s, s' \rangle \mid \text{some property}\}$

# Breadth-first Search with Progression and BDDs

## Progression Breadth-first Search

**def** bfs-progression($V$, $I$, $O$, $\gamma$):
    $goal\_states := models(\gamma)$
    $reached_0 := \{I\}$
    $i := 0$
    **loop**:
        **if** $reached_i \cap goal\_states \neq \emptyset$:
            **return** solution found
        $reached_{i+1} := reached_i \cup apply(reached_i, O)$
        **if** $reached_{i+1} = reached_i$:
            **return** no solution exists
        $i := i + 1$

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Use *bdd-formula*.

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Use *bdd-singleton*.

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Use *bdd-intersection*, *bdd-false*, *bdd-equals*.

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Use *bdd-union*.

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Use *bdd-equals*.

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How to do this?

# The *apply* Function (1)

We need an operation that

- for a set of states *reached* (given as a BDD)
- and a set of operators $O$
- computes the set of states (as a BDD) that result from applying some operator $o \in O$ in some state $s \in \mathit{reached}$.

We have seen something similar already...

# Translating Operators into Formulas

**Definition (Operators in Propositional Logic)**

Let $o$ be an operator and $V$ a set of state variables.

Define $\tau_V(o) := pre(o) \wedge \bigwedge_{v \in V}(regr(v, eff(o)) \leftrightarrow v')$.

States that $o$ is applicable and describes how

- the new value of $v$, represented by $v'$,
- must relate to the old state, described by variables $V$.

# The *apply* Function (2)

- The formula $\tau_V(o)$ describes all transitions $s \xrightarrow{o} s'$
  - induced by a single operator $o$
  - in terms of variables $V$ describing $s$
  - and variables $V'$ describing $s'$.
- The formula $\bigvee_{o \in O} \tau_V(o)$ describes state transitions by any operator in $O$.
- We can translate this formula to a BDD (over variables $V \cup V'$) with *bdd-formula*.
- The resulting BDD is called the transition relation of the planning task, written as $T_V(O)$.

# The *apply* Function (3)

Using the transition relation, we can compute $apply(reached, O)$ as follows:

## The apply function

**def** apply($reached$, $O$):
    $B := T_V(O)$
    $B :=$ *bdd-intersection*$(B, reached)$
    **for each** $v \in V$:
        $B :=$ *bdd-forget*$(B, v)$
    **for each** $v \in V$:
        $B :=$ *bdd-rename*$(B, v', v)$
    **return** $B$

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This describes the set of state pairs $\langle s, s' \rangle$ where $s'$ is a successor of $s$ in terms of variables $V \cup V'$.

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    **return** $B$

This describes the set of state pairs $\langle s, s' \rangle$ where $s'$ is a successor of $s$ and $s \in \mathit{reached}$ in terms of variables $V \cup V'$.

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Thus, *apply* indeed computes the set of successors of *reached* using operators $O$.

# Discussion

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- This completes the discussion of a (basic) symbolic search algorithm for classical planning.
- We ignored the aspect of solution extraction. This needs some extra work, but is not a major challenge.
- In practice, some steps can be performed slightly more efficiently, but these are comparatively minor details.

# Variable Orders

For good performance, we need a good variable ordering.

- Variables that refer to the same state variable before and after operator application ($v$ and $v'$) should be neighbors in the transition relation BDD.

# Extensions

Symbolic search can be extended to...

- regression and bidirectional search:
  this is very easy and often effective
- uniform-cost search:
  requires some work, but not too difficult in principle
- heuristic search:
  requires a heuristic representable as a BDD;
  has not really been shown to outperform blind symbolic search

# Literature


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# Summary

# Summary

- Symbolic search operates on sets of states
  instead of individual states as in explicit-state search.
- State sets and transition relations can be represented
  as BDDs.
- Based on this, we can implement a blind breadth-first search
  in an efficient way.
- A good variable ordering is crucial for performance.