## Discrete Mathematics in Computer Science E5. Syntax and Semantics of Predicate Logic

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#### Discrete Mathematics in Computer Science — E5. Syntax and Semantics of Predicate Logic

## E5.1 Syntax of Predicate Logic

## E5.2 Semantics of Predicate Logic

## E5.1 Syntax of Predicate Logic

## Limits of Propositional Logic

Cannot well be expressed in propositional logic:

- "Everyone who does the exercises passes the exam."
- "If someone with administrator privileges presses 'delete', all data is gone."
- "Everyone has a mother."
- "If someone is the father of some person, the person is his child."

▷ need more expressive logic

→ predicate logic (a.k.a. first-order logic)

German: Prädikatenlogik (erster Stufe)

## Syntax: Building Blocks

Signatures define allowed symbols.
 analogy: atom set A in propositional logic

- Terms are associated with objects by the semantics.
   no analogy in propositional logic
- Formulas are associated with truth values (true or false) by the semantics. analogy: formulas in propositional logic

German: Signatur, Term, Formel

## Signatures: Definition

#### Definition (Signature)

A signature (of predicate logic) is a 4-tuple  $S = \langle V, C, F, P \rangle$  consisting of the following four disjoint sets:

- a finite or countable set  $\mathcal{V}$  of variable symbols
- a finite or countable set C of constant symbols
- ▶ a finite or countable set  $\mathcal{F}$  of function symbols
- a finite or countable set *P* of predicate symbols (or relation symbols)

Every function symbol  $f \in \mathcal{F}$  and predicate symbol  $P \in \mathcal{P}$  has an associated arity  $ar(f), ar(P) \in \mathbb{N}_1$  (number of arguments).

# German: Variablen-, Konstanten-, Funktions-, Prädikat- und Relationssymbole; Stelligkeit

## Signatures: Terminology and Conventions

#### terminology:

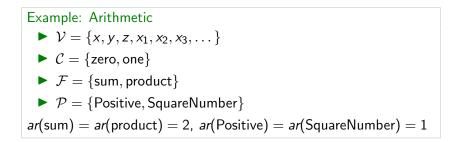
- k-ary (function or predicate) symbol: symbol s with arity ar(s) = k.
- also: unary, binary, ternary

German: k-stellig, unär, binär, ternär

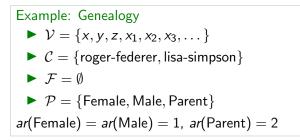
conventions (in this course):

- variable symbols written in *italics*, other symbols upright.
- predicate symbols begin with capital letter, other symbols with lower-case letters

## Signatures: Examples



## Signatures: Examples



## Terms: Definition

Definition (Term) Let  $S = \langle V, C, F, P \rangle$  be a signature. A term (over S) is inductively constructed according to the following rules: • Every variable symbol  $v \in V$  is a term. • Every constant symbol  $c \in C$  is a term.

If t<sub>1</sub>,..., t<sub>k</sub> are terms and f ∈ F is a function symbol with arity k, then f(t<sub>1</sub>,..., t<sub>k</sub>) is a term.

#### German: Term

examples:



#### lisa-simpson

 $\triangleright$  sum(x<sub>3</sub>, product(one, x<sub>5</sub>))

## Formulas: Definition

#### Definition (Formula)

. . .

For a signature  $S = \langle V, C, F, P \rangle$  the set of predicate logic formulas (over S) is inductively defined as follows:

- If t<sub>1</sub>,..., t<sub>k</sub> are terms (over S) and P ∈ P is a k-ary predicate symbol, then the atomic formula (or the atom) P(t<sub>1</sub>,..., t<sub>k</sub>) is a formula over S.
- If t₁ and t₂ are terms (over S), then the identity (t₁ = t₂) is a formula over S.

If x ∈ V is a variable symbol and φ a formula over S, then the universal quantification ∀x φ and the existential quantification ∃x φ are formulas over S.

#### German: atomare Formel, Atom, Identität, Allquantifizierung, Existenzquantifizierung

## Formulas: Definition

#### Definition (Formula)

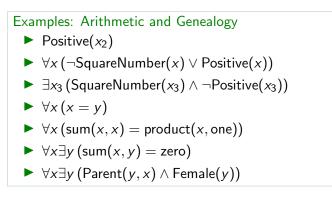
. . .

For a signature  $S = \langle V, C, F, P \rangle$  the set of predicate logic formulas (over S) is inductively defined as follows:

- If  $\varphi$  is a formula over S, then so is its negation  $\neg \varphi$ .
- If φ and ψ are formulas over S, then so are the conjunction (φ ∧ ψ) and the disjunction (φ ∨ ψ).

#### German: Negation, Konjunktion, Disjunktion

## Formulas: Examples



Terminology: The symbols  $\forall$  and  $\exists$  are called quantifiers.

#### German: Quantoren

## Abbreviations and Placement of Parentheses by Convention

abbreviations:

- $(\varphi \to \psi)$  is an abbreviation for  $(\neg \varphi \lor \psi)$ .
- $(\varphi \leftrightarrow \psi)$  is an abbreviation for  $((\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi))$ .
- Sequences of the same quantifier can be abbreviated. For example:
  - $\blacktriangleright \forall x \forall y \forall z \varphi \rightsquigarrow \forall x y z \varphi$
  - $\blacktriangleright \exists x \exists y \exists z \varphi \rightsquigarrow \exists x y z \varphi$
  - $\blacktriangleright \forall w \exists x \exists y \forall z \varphi \rightsquigarrow \forall w \exists xy \forall z \varphi$

placement of parentheses by convention:

- analogous to propositional logic
- quantifiers  $\forall$  and  $\exists$  bind more strongly than anything else.

► example:  $\forall x P(x) \rightarrow Q(x)$  corresponds to  $(\forall x P(x) \rightarrow Q(x))$ , not  $\forall x (P(x) \rightarrow Q(x))$ .

## Exercise

$$\begin{aligned} \mathcal{S} &= \langle \{x,y,z\}, \{c\}, \{f,g,h\}, \{Q,R,S\} \rangle \text{ with} \\ ar(f) &= 3, ar(g) = ar(h) = 1, ar(Q) = 2, ar(R) = ar(S) = 1 \end{aligned}$$

$$\blacktriangleright (g(x) = R(y))$$

$$\blacktriangleright (g(x) = f(y, c, h(x)))$$

$$\blacktriangleright (\mathsf{R}(x) \land \forall x \, \mathsf{S}(x))$$

$$(\forall x \exists y (g(x) = y) \lor (h(x) = c))$$

# Which expressions are syntactically correct formulas or terms for $\mathcal{S}$ ? What kind of term/formula?

## E5.2 Semantics of Predicate Logic

## Semantics: Motivation

- interpretations in propositional logic: truth assignments for the propositional variables
- There are no propositional variables in predicate logic.
- instead: interpretation determines meaning of the constant, function and predicate symbols.
- meaning of variable symbols not determined by interpretation but by separate variable assignment

## Interpretations and Variable Assignments

Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature.

Definition (Interpretation, Variable Assignment)

An interpretation (for S) is a pair  $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  of:

- a non-empty set U called the universe and
- a function <sup>I</sup> that assigns a meaning to the constant, function, and predicate symbols:
  - $c^{\mathcal{I}} \in U$  for constant symbols  $c \in C$
  - $f^{\mathcal{I}}: U^k \to U$  for k-ary function symbols  $f \in \mathcal{F}$
  - $\mathsf{P}^{\mathcal{I}} \subseteq U^k$  for *k*-ary predicate symbols  $\mathsf{P} \in \mathcal{P}$

A variable assignment (for S and universe U) is a function  $\alpha : \mathcal{V} \to U$ .

#### German: Interpretation, Universum (or Grundmenge), Variablenzuweisung

## Interpretations and Variable Assignments: Example

Example signature:  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  with  $\mathcal{V} = \{x, y, z\}$ ,  $C = \{\text{zero, one}\}, \mathcal{F} = \{\text{sum, product}\}, \mathcal{P} = \{\text{SquareNumber}\}$ ar(sum) = ar(product) = 2, ar(SquareNumber) = 1 $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  with  $\blacktriangleright$   $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$  $\blacktriangleright$  zero<sup> $\mathcal{I}$ </sup> =  $\mu_0$ • one<sup> $\mathcal{I}$ </sup> =  $\mu_1$ •  $\operatorname{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \mod 7}$  for all  $i, j \in \{0, \dots, 6\}$ ▶ product<sup> $\mathcal{I}</sup>(u_i, u_i) = u_{(i \cdot i) \mod 7}$  for all  $i, j \in \{0, \dots, 6\}$ </sup> • SquareNumber  $\mathcal{I} = \{u_0, u_1, u_2, u_4\}$  $\alpha = \{ x \mapsto u_5, y \mapsto u_5, z \mapsto u_0 \}$ 

## Semantics: Informally

Example:  $(\forall x (Block(x) \rightarrow Red(x)) \land Block(a))$ "For all objects x: if x is a block, then x is red. Also, the object called a is a block."

- **Terms** are interpreted as objects.
- Unary predicates denote properties of objects (to be a block, to be red, to be a square number, ...).
- General predicates denote relations between objects (to be someone's child, to have a common divisor, ...).
- ► Universally quantified formulas ("∀") are true if they hold for every object in the universe.
- ► Existentially quantified formulas ("∃") are true if they hold for at least one object in the universe.

## Interpretations of Terms

Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature.

#### Definition (Interpretation of a Term)

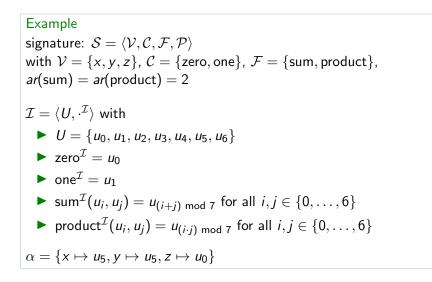
Let  $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  be an interpretation for S, and let  $\alpha$  be a variable assignment for S and universe U. Let t be a term over S. The interpretation of t under  $\mathcal{I}$  and  $\alpha$ , written as  $t^{\mathcal{I},\alpha}$ , is the element of the universe U defined as follows:

► If 
$$t = x$$
 with  $x \in \mathcal{V}$  ( $t$  is a variable term):  
 $x^{\mathcal{I},\alpha} = \alpha(x)$ 

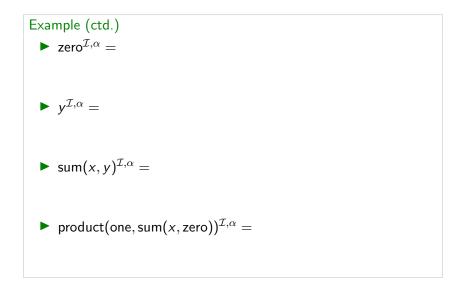
► If 
$$t = c$$
 with  $c \in C$  (*t* is a constant term):  
 $c^{\mathcal{I},\alpha} = c^{\mathcal{I}}$ 

► If 
$$t = f(t_1, ..., t_k)$$
 ( $t$  is a function term):  
 $f(t_1, ..., t_k)^{\mathcal{I}, \alpha} = f^{\mathcal{I}}(t_1^{\mathcal{I}, \alpha}, ..., t_k^{\mathcal{I}, \alpha})$ 

### Interpretations of Terms: Example



#### Interpretations of Terms: Example (ctd.)



### Semantics of Predicate Logic Formulas

Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature.

#### Definition (Formula is Satisfied or True) Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for S, and let $\alpha$ be a variable assignment for S and universe U. We say that $\mathcal{I}$ and $\alpha$ satisfy a predicate logic formula $\varphi$ (also: $\varphi$ is true under $\mathcal{I}$ and $\alpha$ ), written: $\mathcal{I}, \alpha \models \varphi$ ,

according to the following inductive rules:

$$\begin{split} \mathcal{I}, \alpha &\models \mathsf{P}(t_1, \dots, t_k) & \text{iff } \langle t_1^{\mathcal{I}, \alpha}, \dots, t_k^{\mathcal{I}, \alpha} \rangle \in \mathsf{P}^{\mathcal{I}} \\ \mathcal{I}, \alpha &\models (t_1 = t_2) & \text{iff } t_1^{\mathcal{I}, \alpha} = t_2^{\mathcal{I}, \alpha} \\ \mathcal{I}, \alpha &\models \neg \varphi & \text{iff } \mathcal{I}, \alpha \not\models \varphi \\ \mathcal{I}, \alpha &\models (\varphi \land \psi) & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi \\ \mathcal{I}, \alpha &\models (\varphi \lor \psi) & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \end{split}$$

#### German: $\mathcal{I}$ und $\alpha$ erfüllen $\varphi$ (also: $\varphi$ ist wahr unter $\mathcal{I}$ und $\alpha$ )

. . .

## Semantics of Predicate Logic Formulas

Let 
$$\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$$
 be a signature.

```
Definition (Formula is Satisfied or True)
             \begin{split} \mathcal{I}, \alpha \models \forall x \varphi & \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for all } u \in U \\ \mathcal{I}, \alpha \models \exists x \varphi & \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for at least one } u \in U \end{split} 
 where \alpha[x := u] is the same variable assignment as \alpha,
 except that it maps variable x to the value u.
 Formally:
(\alpha[x := u])(z) = \begin{cases} u & \text{if } z = x \\ \alpha(z) & \text{if } z \neq x \end{cases}
```

## Semantics: Example

Example signature:  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with  $\mathcal{V} = \{x, y, z\}, \mathcal{C} = \{a, b\}, \mathcal{F} = \emptyset, \mathcal{P} = \{Block, Red\},\$ ar(Block) = ar(Red) = 1. $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  with  $\blacktriangleright$   $U = \{u_1, u_2, u_3, u_4, u_5\}$  $\blacktriangleright a^{\mathcal{I}} = \mu_1$  $\blacktriangleright$  h<sup> $\mathcal{I}$ </sup> = u<sub>3</sub> ▶ Block<sup> $\mathcal{I}$ </sup> = { $u_1, u_2$ } ▶  $\operatorname{Red}^{\mathcal{I}} = \{u_1, u_2, u_3, u_5\}$  $\alpha = \{ x \mapsto u_1, y \mapsto u_2, z \mapsto u_1 \}$ 

## Semantics: Example (ctd.)

