Discrete Mathematics in Computer Science

E5. Syntax and Semantics of Predicate Logic

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E5.1 Syntax of Predicate Logic

Limits of Propositional Logic

Cannot well be expressed in propositional logic:

- "Everyone who does the exercises passes the exam."
- "If someone with administrator privileges presses 'delete', all data is gone."
- "Everyone has a mother."
- "If someone is the father of some person, the person is his child."

▷ need more expressive logic
$\rightsquigarrow$ predicate logic (a.k.a. first-order logic)

German: Prädikatenlogik (erster Stufe)

Syntax: Building Blocks

- Signatures define allowed symbols.
  analogy: atom set A in propositional logic
- Terms are associated with objects by the semantics.
  no analogy in propositional logic
- Formulas are associated with truth values (true or false) by the semantics.
  analogy: formulas in propositional logic

German: Signatur, Term, Formel

Signatures: Definition

Definition (Signature)

A signature (of predicate logic) is a 4-tuple $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ consisting of the following four disjoint sets:

- a finite or countable set $\mathcal{V}$ of variable symbols
- a finite or countable set $\mathcal{C}$ of constant symbols
- a finite or countable set $\mathcal{F}$ of function symbols
- a finite or countable set $\mathcal{P}$ of predicate symbols (or relation symbols)

Every function symbol $\mathsf{f} \in \mathcal{F}$ and predicate symbol $\mathsf{P} \in \mathcal{P}$ has an associated arity $ar(\mathsf{f}), ar(\mathsf{P}) \in \mathbb{N}_1$ (number of arguments).

German: Variablen-, Konstanten-, Funktions-, Prädikat- und Relationssymbole; Stelligkeit

Signatures: Terminology and Conventions

terminology:

- *k*-ary (function or predicate) symbol:
  symbol s with arity $ar(\mathrm{s}) = k$.
- also: unary, binary, ternary

German: *k*-stellig, unär, binär, ternär

conventions (in this course):

- variable symbols written in *italics*,
  other symbols upright.
- predicate symbols begin with capital letter,
  other symbols with lower-case letters

Signatures: Examples

Example: Arithmetic

- $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- $\mathcal{C} = \{\text{zero}, \text{one}\}$
- $\mathcal{F} = \{\text{sum}, \text{product}\}$
- $\mathcal{P} = \{\text{Positive}, \text{SquareNumber}\}$

$ar(\text{sum}) = ar(\text{product}) = 2$, $ar(\text{Positive}) = ar(\text{SquareNumber}) = 1$

Signatures: Examples

Example: Genealogy

- $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- $\mathcal{C} = \{\text{roger-federer}, \text{lisa-simpson}\}$
- $\mathcal{F} = \emptyset$
- $\mathcal{P} = \{\text{Female}, \text{Male}, \text{Parent}\}$

$ar(\text{Female}) = ar(\text{Male}) = 1$, $ar(\text{Parent}) = 2$

Terms: Definition

Definition (Term)

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.
A term (over $\mathcal{S}$) is inductively constructed according to the following rules:

- Every variable symbol $v \in \mathcal{V}$ is a term.
- Every constant symbol $\mathsf{c} \in \mathcal{C}$ is a term.
- If $t_1, \dots, t_k$ are terms and $\mathsf{f} \in \mathcal{F}$ is a function symbol with arity k, then $\mathsf{f}(t_1, \dots, t_k)$ is a term.

German: Term

examples:

- x_4
- lisa-simpson
- $\mathsf{sum}(x_3, \mathsf{product}(\mathsf{one}, x_5))$

Formulas: Definition

Definition (Formula)

For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas (over $\mathcal{S}$) is inductively defined as follows:

- If $t_1, \dots, t_k$ are terms (over $\mathcal{S}$) and $\mathsf{P} \in \mathcal{P}$ is a k-ary predicate symbol, then the atomic formula (or the atom) $\mathsf{P}(t_1, \dots, t_k)$ is a formula over $\mathcal{S}$.
- If t_1 and t_2 are terms (over $\mathcal{S}$), then the identity $(t_1 = t_2)$ is a formula over $\mathcal{S}$.
- If $x \in \mathcal{V}$ is a variable symbol and φ a formula over $\mathcal{S}$, then the universal quantification $\forall x\, \varphi$ and the existential quantification $\exists x\, \varphi$ are formulas over $\mathcal{S}$.

...

German: atomare Formel, Atom, Identität, Allquantifizierung, Existenzquantifizierung

Formulas: Definition

Definition (Formula)

For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas (over $\mathcal{S}$) is inductively defined as follows:

...

- If φ is a formula over $\mathcal{S}$, then so is its negation $\neg\varphi$.
- If φ and ψ are formulas over $\mathcal{S}$, then so are the conjunction $(\varphi \wedge \psi)$ and the disjunction $(\varphi \vee \psi)$.

German: Negation, Konjunktion, Disjunktion

Formulas: Examples

Examples: Arithmetic and Genealogy

- $\mathsf{Positive}(x_2)$
- $\forall x\,(\neg\mathsf{SquareNumber}(x) \lor \mathsf{Positive}(x))$
- $\exists x_3\,(\mathsf{SquareNumber}(x_3) \land \neg\mathsf{Positive}(x_3))$
- $\forall x\,(x = y)$
- $\forall x\,(\mathsf{sum}(x, x) = \mathsf{product}(x, \mathsf{one}))$
- $\forall x \exists y\,(\mathsf{sum}(x, y) = \mathsf{zero})$
- $\forall x \exists y\,(\mathsf{Parent}(y, x) \land \mathsf{Female}(y))$

Terminology: The symbols $\forall$ and $\exists$ are called quantifiers.

German: Quantoren

Abbreviations and Placement of Parentheses by Convention

abbreviations:

- $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$.
- $(\varphi \leftrightarrow \psi)$ is an abbreviation for $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.
- Sequences of the same quantifier can be abbreviated. For example:
  - $\forall x \forall y \forall z\, \varphi \rightsquigarrow \forall xyz\, \varphi$
  - $\exists x \exists y \exists z\, \varphi \rightsquigarrow \exists xyz\, \varphi$
  - $\forall w \exists x \exists y \forall z\, \varphi \rightsquigarrow \forall w \exists xy \forall z\, \varphi$

placement of parentheses by convention:

- analogous to propositional logic
- quantifiers $\forall$ and $\exists$ bind more strongly than anything else.
- example: $\forall x\, \mathrm{P}(x) \rightarrow \mathrm{Q}(x)$ corresponds to $(\forall x\, \mathrm{P}(x) \rightarrow \mathrm{Q}(x))$, not $\forall x\, (\mathrm{P}(x) \rightarrow \mathrm{Q}(x))$.

Exercise

$\mathcal{S} = \langle \{x, y, z\}, \{\mathsf{c}\}, \{\mathsf{f}, \mathsf{g}, \mathsf{h}\}, \{\mathsf{Q}, \mathsf{R}, \mathsf{S}\} \rangle$ with
$ar(\mathsf{f}) = 3, ar(\mathsf{g}) = ar(\mathsf{h}) = 1, ar(\mathsf{Q}) = 2, ar(\mathsf{R}) = ar(\mathsf{S}) = 1$

- $\mathsf{f}(x, y)$
- $(\mathsf{g}(x) = \mathsf{R}(y))$
- $(\mathsf{g}(x) = \mathsf{f}(y, \mathsf{c}, \mathsf{h}(x)))$
- $(\mathsf{R}(x) \land \forall x\, \mathsf{S}(x))$
- $\forall \mathsf{c}\, \mathsf{Q}(\mathsf{c}, x)$
- $(\forall x \exists y\, (\mathsf{g}(x) = y) \lor (\mathsf{h}(x) = \mathsf{c}))$

Which expressions are syntactically correct formulas or terms for $\mathcal{S}$?
What kind of term/formula?

E5.2 Semantics of Predicate Logic

Semantics: Motivation

- interpretations in propositional logic:
  truth assignments for the propositional variables
- There are no propositional variables in predicate logic.
- instead: interpretation determines meaning
  of the constant, function and predicate symbols.
- meaning of variable symbols not determined by interpretation
  but by separate variable assignment

Interpretations and Variable Assignments

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation, Variable Assignment)

An interpretation (for $\mathcal{S}$) is a pair $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ of:

- a non-empty set U called the universe and
- a function $\cdot^{\mathcal{I}}$ that assigns a meaning to the constant, function, and predicate symbols:
  - $\mathsf{c}^{\mathcal{I}} \in U$ for constant symbols $\mathsf{c} \in \mathcal{C}$
  - $\mathsf{f}^{\mathcal{I}} : U^k \to U$ for k-ary function symbols $\mathsf{f} \in \mathcal{F}$
  - $\mathsf{P}^{\mathcal{I}} \subseteq U^k$ for k-ary predicate symbols $\mathsf{P} \in \mathcal{P}$

A variable assignment (for $\mathcal{S}$ and universe U) is a function $\alpha : \mathcal{V} \to U$.

German: Interpretation, Universum (or Grundmenge), Variablenzuweisung

Interpretations and Variable Assignments: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{V} = \{x, y, z\}$,
$\mathcal{C} = \{\text{zero}, \text{one}\}$, $\mathcal{F} = \{\text{sum}, \text{product}\}$, $\mathcal{P} = \{\text{SquareNumber}\}$
$ar(\text{sum}) = ar(\text{product}) = 2$, $ar(\text{SquareNumber}) = 1$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- $\text{zero}^{\mathcal{I}} = u_0$
- $\text{one}^{\mathcal{I}} = u_1$
- $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- $\text{SquareNumber}^{\mathcal{I}} = \{u_0, u_1, u_2, u_4\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Semantics: Informally

Example: $(\forall x(\mathrm{Block}(x) \rightarrow \mathrm{Red}(x)) \land \mathrm{Block}(\mathrm{a}))$
"For all objects x: if x is a block, then x is red.
Also, the object called a is a block."

- Terms are interpreted as objects.
- Unary predicates denote properties of objects (to be a block, to be red, to be a square number, ...).
- General predicates denote relations between objects (to be someone's child, to have a common divisor, ...).
- Universally quantified formulas ("$\forall$") are true if they hold for every object in the universe.
- Existentially quantified formulas ("$\exists$") are true if they hold for at least one object in the universe.

Interpretations of Terms

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation of a Term)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for $\mathcal{S}$,
and let α be a variable assignment for $\mathcal{S}$ and universe U.

Let t be a term over $\mathcal{S}$.
The interpretation of t under $\mathcal{I}$ and α, written as $t^{\mathcal{I},\alpha}$,
is the element of the universe U defined as follows:

- If $t = x$ with $x \in \mathcal{V}$ (t is a variable term):
  $x^{\mathcal{I},\alpha} = \alpha(x)$
- If $t = \mathsf{c}$ with $\mathsf{c} \in \mathcal{C}$ (t is a constant term):
  $\mathsf{c}^{\mathcal{I},\alpha} = \mathsf{c}^{\mathcal{I}}$
- If $t = \mathsf{f}(t_1, \dots, t_k)$ (t is a function term):
  $\mathsf{f}(t_1, \dots, t_k)^{\mathcal{I},\alpha} = \mathsf{f}^{\mathcal{I}}(t_1^{\mathcal{I},\alpha}, \dots, t_k^{\mathcal{I},\alpha})$

Interpretations of Terms: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$
with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\mathsf{zero}, \mathsf{one}\}$, $\mathcal{F} = \{\mathsf{sum}, \mathsf{product}\}$,
$ar(\mathsf{sum}) = ar(\mathsf{product}) = 2$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- $\mathsf{zero}^{\mathcal{I}} = u_0$
- $\mathsf{one}^{\mathcal{I}} = u_1$
- $\mathsf{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- $\mathsf{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Interpretations of Terms: Example (ctd.)

Example (ctd.)

- $\mathsf{zero}^{\mathcal{I},\alpha} =$
- $y^{\mathcal{I},\alpha} =$
- $\mathsf{sum}(x, y)^{\mathcal{I},\alpha} =$
- $\mathsf{product}(\mathsf{one}, \mathsf{sum}(x, \mathsf{zero}))^{\mathcal{I},\alpha} =$

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for $\mathcal{S}$,
and let α be a variable assignment for $\mathcal{S}$ and universe U.
We say that $\mathcal{I}$ and α **satisfy** a predicate logic formula φ
(also: φ is **true** under $\mathcal{I}$ and α), written: $\mathcal{I}, \alpha \models \varphi$,
according to the following inductive rules:

$$
\begin{aligned}
\mathcal{I}, \alpha \models \mathsf{P}(t_1, \dots, t_k) \quad & \text{iff } \langle t_1^{\mathcal{I},\alpha}, \dots, t_k^{\mathcal{I},\alpha} \rangle \in \mathsf{P}^{\mathcal{I}} \\
\mathcal{I}, \alpha \models (t_1 = t_2) \quad & \text{iff } t_1^{\mathcal{I},\alpha} = t_2^{\mathcal{I},\alpha} \\
\mathcal{I}, \alpha \models \neg\varphi \quad & \text{iff } \mathcal{I}, \alpha \not\models \varphi \\
\mathcal{I}, \alpha \models (\varphi \land \psi) \quad & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models (\varphi \lor \psi) \quad & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \qquad \dots
\end{aligned}
$$

German: $\mathcal{I}$ und α erfüllen φ (also: φ ist wahr unter $\mathcal{I}$ und α)

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

...

$$\mathcal{I}, \alpha \models \forall x \varphi \quad \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for all } u \in U$$

$$\mathcal{I}, \alpha \models \exists x \varphi \quad \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for at least one } u \in U$$

where $\alpha[x := u]$ is the same variable assignment as α, except that it maps variable x to the value u.
Formally:

$$(\alpha[x := u])(z) = \begin{cases} u & \text{if } z = x \\ \alpha(z) & \text{if } z \neq x \end{cases}$$

Semantics: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$
with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\mathsf{a}, \mathsf{b}\}$, $\mathcal{F} = \emptyset$, $\mathcal{P} = \{\mathsf{Block}, \mathsf{Red}\}$,
$ar(\mathsf{Block}) = ar(\mathsf{Red}) = 1$.

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_1, u_2, u_3, u_4, u_5\}$
- $\mathsf{a}^{\mathcal{I}} = u_1$
- $\mathsf{b}^{\mathcal{I}} = u_3$
- $\mathsf{Block}^{\mathcal{I}} = \{u_1, u_2\}$
- $\mathsf{Red}^{\mathcal{I}} = \{u_1, u_2, u_3, u_5\}$

$\alpha = \{x \mapsto u_1, y \mapsto u_2, z \mapsto u_1\}$

Semantics: Example (ctd.)

Example (ctd.)

Questions:

- $\mathcal{I}, \alpha \models (\mathsf{Block}(\mathsf{b}) \lor \lnot\mathsf{Block}(\mathsf{b}))$?
- $\mathcal{I}, \alpha \models (\mathsf{Block}(x) \rightarrow (\mathsf{Block}(x) \lor \lnot\mathsf{Block}(y)))$?
- $\mathcal{I}, \alpha \models (\mathsf{Block}(\mathsf{a}) \land \mathsf{Block}(\mathsf{b}))$?
- $\mathcal{I}, \alpha \models \forall x(\mathsf{Block}(x) \rightarrow \mathsf{Red}(x))$?