Discrete Mathematics in Computer Science

Inference Rules and Calculi

Malte Helmert, Gabriele Röger

University of Basel

Inference: Motivation

- up to now: proof of logical consequence with semantic arguments
- no general algorithm
- solution: produce formulas that are logical consequences of given formulas with syntactic inference rules
- advantage: mechanical method that can easily be implemented as an algorithm

Inference Rules

- Inference rules have the form

$$\frac{\varphi_1, \dots, \varphi_k}{\psi}.$$

- Meaning: "Every model of $\varphi_1, \dots, \varphi_k$ is a model of ψ."
- An axiom is an inference rule with $k = 0$.
- A set of inference rules is called a calculus or proof system.

German: Inferenzregel, Axiom, (der) Kalkül, Beweissystem

Some Inference Rules for Propositional Logic

Modus ponens $\dfrac{\varphi,\ (\varphi \to \psi)}{\psi}$

Modus tollens $\dfrac{\neg\psi,\ (\varphi \to \psi)}{\neg\varphi}$

$\wedge$-elimination $\dfrac{(\varphi \wedge \psi)}{\varphi}$ $\dfrac{(\varphi \wedge \psi)}{\psi}$

$\wedge$-introduction $\dfrac{\varphi,\ \psi}{(\varphi \wedge \psi)}$

$\vee$-introduction $\dfrac{\varphi}{(\varphi \vee \psi)}$

$\leftrightarrow$-elimination $\dfrac{(\varphi \leftrightarrow \psi)}{(\varphi \to \psi)}$ $\dfrac{(\varphi \leftrightarrow \psi)}{(\psi \to \varphi)}$

Derivation

Definition (Derivation)

A derivation or proof of a formula φ from a knowledge base KB is a sequence of formulas $\psi_1, \ldots, \psi_k$ with

- $\psi_k = \varphi$ and
- for all $i \in \{1, \ldots, k\}$:
  - $\psi_i \in \mathrm{KB}$, or
  - ψ_i is the result of the application of an inference rule to elements from $\{\psi_1, \ldots, \psi_{i-1}\}$.

German: Ableitung, Beweis

Derivation: Example

Example

Given: $\text{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

Derivation: Example

Example

Given: $\text{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \land R) \rightarrow S)\}$

Task: Find derivation of $(S \land R)$ from KB.

1. P (KB)

Derivation: Example

Example

Given: $\text{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)

Derivation: Example

Example

Given: $\text{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$
Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)

Derivation: Example

Example

Given: $\text{KB} = \{P, (P \to Q), (P \to R), ((Q \wedge R) \to S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \to Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \to R)$ (KB)
5. R (1, 4, Modus ponens)

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)
5. R (1, 4, Modus ponens)
6. $(Q \wedge R)$ (3, 5, $\wedge$-introduction)

Derivation: Example

Example

Given: $\text{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)
5. R (1, 4, Modus ponens)
6. $(Q \wedge R)$ (3, 5, $\wedge$-introduction)
7. $((Q \wedge R) \rightarrow S)$ (KB)

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)
5. R (1, 4, Modus ponens)
6. $(Q \wedge R)$ (3, 5, $\wedge$-introduction)
7. $((Q \wedge R) \rightarrow S)$ (KB)
8. S (6, 7, Modus ponens)

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)
5. R (1, 4, Modus ponens)
6. $(Q \wedge R)$ (3, 5, $\wedge$-introduction)
7. $((Q \wedge R) \rightarrow S)$ (KB)
8. S (6, 7, Modus ponens)
9. $(S \wedge R)$ (8, 5, $\wedge$-introduction)

Correctness and Completeness

Definition (Correctness and Completeness of a Calculus)

We write $\mathrm{KB} \vdash_C \varphi$ if there is a derivation of φ from KB in calculus C.
(If calculus C is clear from context, also only $\mathrm{KB} \vdash \varphi$.)

A calculus C is correct if for all KB and φ
$\mathrm{KB} \vdash_C \varphi$ implies $\mathrm{KB} \models \varphi$.

A calculus C is complete if for all KB and φ
$\mathrm{KB} \models \varphi$ implies $\mathrm{KB} \vdash_C \varphi$.

Correctness and Completeness

Definition (Correctness and Completeness of a Calculus)

We write $\mathrm{KB} \vdash_C \varphi$ if there is a derivation of φ from KB in calculus C.
(If calculus C is clear from context, also only $\mathrm{KB} \vdash \varphi$.)

A calculus C is correct if for all KB and φ
$\mathrm{KB} \vdash_C \varphi$ implies $\mathrm{KB} \models \varphi$.

A calculus C is complete if for all KB and φ
$\mathrm{KB} \models \varphi$ implies $\mathrm{KB} \vdash_C \varphi$.

Consider calculus C, consisting of the derivation rules seen earlier.
Question: Is C correct?
Question: Is C complete?

German: korrekt, vollständig

Refutation-completeness

- We obviously want correct calculi.
- Do we always need a complete calculus?

German: widerlegungsvollständig

Refutation-completeness

- We obviously want correct calculi.
- Do we always need a complete calculus?
- Contradiction theorem:
  $\mathrm{KB} \cup \{\varphi\}$ is unsatisfiable iff $\mathrm{KB} \models \neg\varphi$
- This implies that $\mathrm{KB} \models \varphi$ iff $\mathrm{KB} \cup \{\neg\varphi\}$ is unsatisfiable.
- We can reduce the general implication problem
  to a test of unsatisfiability.

German: widerlegungsvollständig

Refutation-completeness

- We obviously want correct calculi.
- Do we always need a complete calculus?
- Contradiction theorem:
  KB $\cup \{\varphi\}$ is unsatisfiable iff KB $\models \neg\varphi$
- This implies that KB $\models \varphi$ iff KB $\cup \{\neg\varphi\}$ is unsatisfiable.
- We can reduce the general implication problem
  to a test of unsatisfiability.
- In calculi, we use the special symbol $\square$ for (provably) unsatisfiable formulas.

German: widerlegungsvollständig

Refutation-completeness

- We obviously want correct calculi.
- Do we always need a complete calculus?
- Contradiction theorem:
  $\mathrm{KB} \cup \{\varphi\}$ is unsatisfiable iff $\mathrm{KB} \models \neg\varphi$
- This implies that $\mathrm{KB} \models \varphi$ iff $\mathrm{KB} \cup \{\neg\varphi\}$ is unsatisfiable.
- We can reduce the general implication problem to a test of unsatisfiability.
- In calculi, we use the special symbol $\square$ for (provably) unsatisfiable formulas.

Definition (Refutation-Completeness)

A calculus C is refutation-complete if $\mathrm{KB} \vdash_C \square$ for all unsatisfiable KB.

German: widerlegungsvollständig

Discrete Mathematics in Computer Science

Resolution Calculus

Malte Helmert, Gabriele Röger

University of Basel

Resolution: Idea

- Resolution is a refutation-complete calculus
  for knowledge bases in conjunctive normal form.

Resolution: Idea

- Resolution is a refutation-complete calculus for knowledge bases in conjunctive normal form.
- Every knowledge base can be transformed into equivalent formulas in CNF.
  - Transformation can require exponential time.
  - Alternative: efficient transformation into equisatisfiable formulas (not part of this course)

Resolution: Idea

- Resolution is a refutation-complete calculus for knowledge bases in conjunctive normal form.
- Every knowledge base can be transformed into equivalent formulas in CNF.
  - Transformation can require exponential time.
  - Alternative: efficient transformation into equisatisfiable formulas (not part of this course)
- Show $\mathrm{KB} \models \varphi$ by deriving $\mathrm{KB} \cup \{\neg\varphi\} \vdash_R \square$ with resolution calculus R.

Resolution: Idea

- Resolution is a refutation-complete calculus for knowledge bases in conjunctive normal form.
- Every knowledge base can be transformed into equivalent formulas in CNF.
  - Transformation can require exponential time.
  - Alternative: efficient transformation into equisatisfiable formulas (not part of this course)
- Show $\mathrm{KB} \models \varphi$ by deriving $\mathrm{KB} \cup \{\neg\varphi\} \vdash_R \square$ with resolution calculus R.
- Resolution can require exponential time.
- This is probably the case for all refutation-complete proof methods. $\rightsquigarrow$ complexity theory

German: Resolution, erfüllbarkeitsäquivalent

Knowledge Base as Set of Clauses

Simplified notation of knowledge bases in CNF

- Formula in CNF as set of clauses
  (due to commutativity, idempotence, associativity of $\wedge$)
- Set of formulas as set of clauses
- Clause as set of literals
  (due to commutativity, idempotence, associativity of $\vee$)
- Knowledge base as set of sets of literals

Knowledge Base as Set of Clauses

Simplified notation of knowledge bases in CNF

- Formula in CNF as set of clauses
  (due to commutativity, idempotence, associativity of $\wedge$)
- Set of formulas as set of clauses
- Clause as set of literals
  (due to commutativity, idempotence, associativity of $\vee$)
- Knowledge base as set of sets of literals

Example

$$\mathrm{KB} = \{(P \vee P), ((\neg P \vee Q) \wedge (\neg P \vee R) \wedge (Q \vee \neg P) \wedge R), ((\neg Q \vee \neg R \vee S) \wedge P)\}$$

as set of clauses:

Knowledge Base as Set of Clauses

Simplified notation of knowledge bases in CNF

- Formula in CNF as set of clauses
  (due to commutativity, idempotence, associativity of $\wedge$)
- Set of formulas as set of clauses
- Clause as set of literals
  (due to commutativity, idempotence, associativity of $\vee$)
- Knowledge base as set of sets of literals

Example

$$\text{KB} = \{(P \vee P), ((\neg P \vee Q) \wedge (\neg P \vee R) \wedge (Q \vee \neg P) \wedge R), \\ ((\neg Q \vee \neg R \vee S) \wedge P)\}$$

as set of clauses:

$$\Delta = \{\{P\}, \{\neg P, Q\}, \{\neg P, R\}, \{R\}, \{\neg Q, \neg R, S\}\}$$

Resolution Rule

The resolution calculus consists of a single rule, called resolution rule:

$$\frac{C_1 \cup \{X\},\ C_2 \cup \{\neg X\}}{C_1 \cup C_2},$$

where C_1 and C_2 are (possibly empty) clauses and X is an atomic proposition.

Resolution Rule

The resolution calculus consists of a single rule, called resolution rule:

$$\frac{C_1 \cup \{X\},\ \ C_2 \cup \{\neg X\}}{C_1 \cup C_2},$$

where C_1 and C_2 are (possibly empty) clauses and X is an atomic proposition.

If we derive the empty clause, we write $\Box$ instead of $\{\}$.

Resolution Rule

The resolution calculus consists of a single rule, called resolution rule:

$$\frac{C_1 \cup \{X\},\;\; C_2 \cup \{\neg X\}}{C_1 \cup C_2},$$

where C_1 and C_2 are (possibly empty) clauses and X is an atomic proposition.

If we derive the empty clause, we write $\Box$ instead of $\{\}$.

Terminology:

- X and $\neg X$ are the resolution literals,
- $C_1 \cup \{X\}$ and $C_2 \cup \{\neg X\}$ are the parent clauses, and
- $C_1 \cup C_2$ is the resolvent.

German: Resolutionskalkül, Resolutionsregel, Resolutionsliterale, Elternklauseln, Resolvent

Proof by Resolution

Definition (Proof by Resolution)

A proof by resolution of a clause D from a knowledge base Δ is a sequence of clauses $C_1, \ldots, C_n$ with

- $C_n = D$ and
- for all $i \in \{1, \ldots, n\}$:
  - $C_i \in \Delta$, or
  - C_i is resolvent of two clauses from $\{C_1, \ldots, C_{i-1}\}$.

If there is a proof of D by resolution from Δ, we say that D can be derived with resolution from Δ and write $\Delta \vdash_R D$.

Remark: Resolution is a correct, refutation-complete, but incomplete calculus.

German: Resolutionsbeweis, mit Resolution aus Δ abgeleitet

Proof by Resolution: Example

Proof by Resolution for Testing a Logical Consequence: Example

Given: $\mathrm{KB} = \{P, (P \rightarrow (Q \wedge R))\}$.
Show with resolution that $\mathrm{KB} \models (R \vee S)$.

Proof by Resolution: Example

Proof by Resolution for Testing a Logical Consequence: Example

Given: $\mathrm{KB} = \{P, (P \rightarrow (Q \wedge R))\}$.
Show with resolution that $\mathrm{KB} \models (R \vee S)$.

Three steps:

1. Reduce logical consequence to unsatisfiability.
2. Transform knowledge base into clause form (CNF).
3. Derive empty clause $\square$ with resolution.

Proof by Resolution: Example

Proof by Resolution for Testing a Logical Consequence: Example

Given: $\mathrm{KB} = \{P, (P \to (Q \wedge R))\}$.
Show with resolution that $\mathrm{KB} \models (R \vee S)$.

Three steps:

1. Reduce logical consequence to unsatisfiability.
2. Transform knowledge base into clause form (CNF).
3. Derive empty clause $\square$ with resolution.

Step 1: Reduce logical consequence to unsatisfiability.

Proof by Resolution: Example

Proof by Resolution for Testing a Logical Consequence: Example

Given: $\mathrm{KB} = \{P, (P \rightarrow (Q \wedge R))\}$.
Show with resolution that $\mathrm{KB} \models (R \vee S)$.

Three steps:

1. Reduce logical consequence to unsatisfiability.
2. Transform knowledge base into clause form (CNF).
3. Derive empty clause $\square$ with resolution.

Step 1: Reduce logical consequence to unsatisfiability.

$\mathrm{KB} \models (R \vee S)$ iff $\mathrm{KB} \cup \{\neg(R \vee S)\}$ is unsatisfiable.

Thus, consider
$\mathrm{KB}' = \mathrm{KB} \cup \{\neg(R \vee S)\} = \{P, (P \rightarrow (Q \wedge R)), \neg(R \vee S)\}$.

...

Proof by Resolution: Example (continued)

Proof by Resolution for Testing a Logical Consequence: Example

$\text{KB}' = \{P, (P \rightarrow (Q \land R)), \neg(R \lor S)\}$.

Step 2: Transform knowledge base into clause form (CNF).

Proof by Resolution: Example (continued)

Proof by Resolution for Testing a Logical Consequence: Example

$\text{KB}' = \{P, (P \rightarrow (Q \wedge R)), \neg(R \vee S)\}$.

Step 2: Transform knowledge base into clause form (CNF).

- P
  $\rightsquigarrow$ Clauses:$\{P\}$
- $P \rightarrow (Q \wedge R)) \equiv (\neg P \vee (Q \wedge R)) \equiv ((\neg P \vee Q) \wedge (\neg P \vee R))$
  $\rightsquigarrow$ Clauses:$\{\neg P, Q\}, \{\neg P, R\}$
- $\neg(R \vee S) \equiv (\neg R \wedge \neg S)$
  $\rightsquigarrow$ Clauses:$\{\neg R\}, \{\neg S\}$

Proof by Resolution: Example (continued)

Proof by Resolution for Testing a Logical Consequence: Example

$\text{KB}' = \{P, (P \rightarrow (Q \wedge R)), \neg(R \vee S)\}$.

Step 2: Transform knowledge base into clause form (CNF).

- P
  $\rightsquigarrow$ Clauses:$\{P\}$
- $P \rightarrow (Q \wedge R)) \equiv (\neg P \vee (Q \wedge R)) \equiv ((\neg P \vee Q) \wedge (\neg P \vee R))$
  $\rightsquigarrow$ Clauses:$\{\neg P, Q\}, \{\neg P, R\}$
- $\neg(R \vee S) \equiv (\neg R \wedge \neg S)$
  $\rightsquigarrow$ Clauses:$\{\neg R\}, \{\neg S\}$

$\Delta = \{\{P\}, \{\neg P, Q\}, \{\neg P, R\}, \{\neg R\}, \{\neg S\}\}$

...

Proof by Resolution: Example (continued)

Proof by Resolution for Testing a Logical Consequence: Example

$\Delta = \{\{P\}, \{\neg P, Q\}, \{\neg P, R\}, \{\neg R\}, \{\neg S\}\}$

Step 3: Derive empty clause $\square$ with resolution.

- $C_1 = \{P\}$ (from Δ)
- $C_2 = \{\neg P, Q\}$ (from Δ)
- $C_3 = \{\neg P, R\}$ (from Δ)
- $C_4 = \{\neg R\}$ (from Δ)
- $C_5 = \{Q\}$ (from C_1 and C_2)
- $C_6 = \{\neg P\}$ (from C_3 and C_4)
- $C_7 = \square$ (from C_1 and C_6)

Note: There are shorter proofs. (For example?)

Another Example

Another Example for Resolution

Show with resolution, that KB $\models$ DrinkBeer, where

$$\begin{aligned}\mathsf{KB} = \{&(\neg\mathsf{DrinkBeer} \rightarrow \mathsf{EatFish}),\\ &((\mathsf{EatFish} \wedge \mathsf{DrinkBeer}) \rightarrow \neg\mathsf{EatIceCream}),\\ &((\mathsf{EatIceCream} \vee \neg\mathsf{DrinkBeer}) \rightarrow \neg\mathsf{EatFish})\}.\end{aligned}$$

Proving that Something Does Not Follow

- We can now use resolution proofs to mechanically show KB $\models \varphi$ whenever a given knowledge base logically implies φ.
- Question: How can we use the same mechanism to show that something does not follow (KB $\not\models \varphi$)?