### Discrete Mathematics in Computer Science E2. Properties of Formulas and Equivalences

Malte Helmert, Gabriele Röger

University of Basel

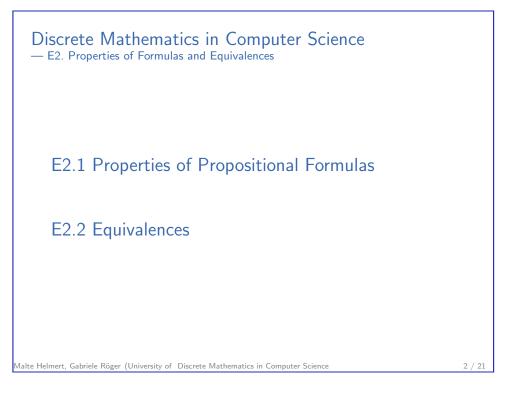
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E2. Properties of Formulas and Equivalences

Properties of Propositional Formulas

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# E2.1 Properties of Propositional Formulas



E2. Properties of Formulas and Equivalences

The Story So Far

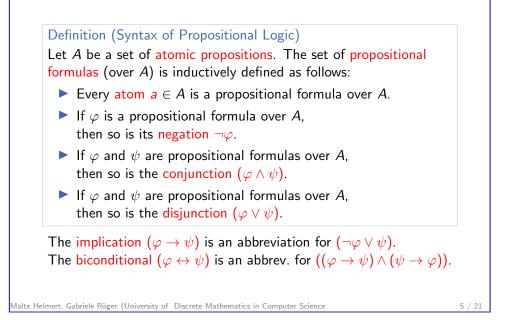
- propositional logic based on atomic propositions
- syntax: which formulas are well-formed?
- semantics: when is a formula true?
- interpretations: important basis of semantics

Properties of Propositional Formulas

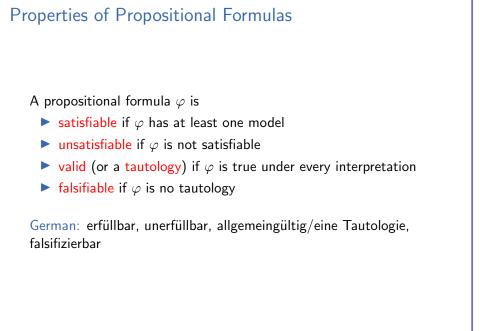


Properties of Propositional Formulas

## Reminder: Syntax of Propositional Logic



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### Reminder: Semantics of Propositional Logic

Properties of Propositional Formulas

Definition (Semantics of Propositional Logic)

A truth assignment (or interpretation) for a set of atomic propositions A is a function  $\mathcal{I} : A \to \{0, 1\}$ .

A propositional formula  $\varphi$  (over A) holds under  $\mathcal{I}$ (written as  $\mathcal{I} \models \varphi$ ) according to the following definition:

$\mathcal{I}\models a$	iff	$\mathcal{I}(a) = 1$	(for $a \in A$ )
$\mathcal{I} \models \neg \varphi$	iff	not $\mathcal{I}\models arphi$	
$\mathcal{I} \models (\varphi \land \psi)$	iff	$\mathcal{I}\models arphi$ and $\mathcal{I}\models \psi$	
$\mathcal{I} \models (\varphi \lor \psi)$	iff	$\mathcal{I}\models\varphi \text{ or }\mathcal{I}\models\psi$	

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Examples

How can we show that a formula has one of these properties?

- Show that  $(A \land B)$  is satisfiable.  $\mathcal{I} = \{A \mapsto 1, B \mapsto 1\}$  (+ simple proof that  $\mathcal{I} \models (A \land B)$ )
- Show that  $(A \land B)$  is falsifiable.  $\mathcal{I} = \{A \mapsto 0, B \mapsto 1\} \ (+ \text{ simple proof that } \mathcal{I} \not\models (A \land B))$
- Show that  $(A \land B)$  is not valid. Follows directly from falsifiability.
- Show that  $(A \land B)$  is not unsatisfiable. Follows directly from satisfiability.

So far all proofs by specifying one interpretation.

How to prove that a given formula is valid/unsatisfiable/ not satisfiable/not falsifiable?

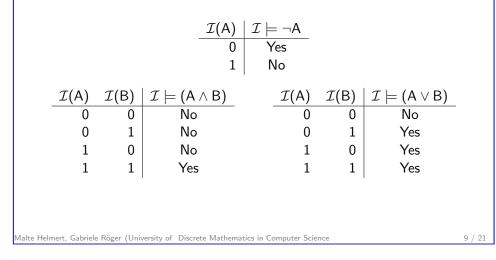
→ must consider all possible interpretations



Properties of Propositional Formulas

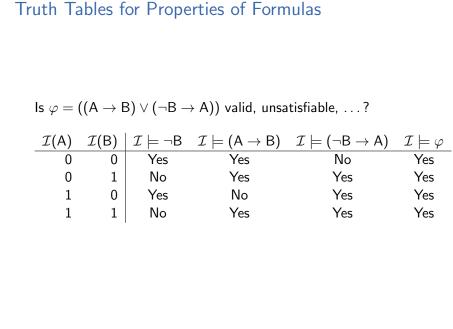
### Truth Tables

Evaluate for all possible interpretations if they are models of the considered formula.



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Properties of Propositional Formulas

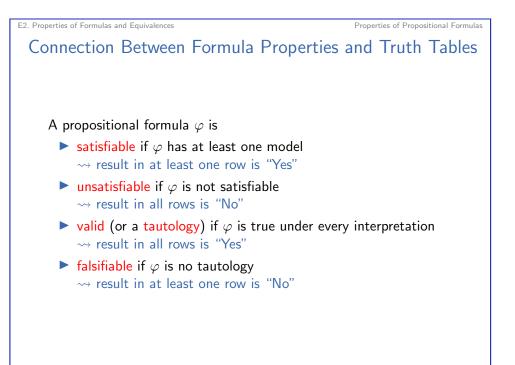


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### Truth Tables in General

Similarly in the case where we consider a formula whose building blocks are themselves arbitrary unspecified formulas:

	$\mathcal{I}\models\varphi$	$\mathcal{I} \models \psi$	$\mathcal{I} \models (\varphi \land \psi)$		
	No	No	No	-	
	No	Yes	No		
	Yes	No	No		
	Yes	Yes	Yes		
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E2. Properties of Formulas and Equivalences	Properties of Propositional F	Formulas	E2. Properties of Formulas and Equivalences
Main Disadvantage of Truth Ta	bles		
How big is a truth table with <i>n</i> atom	ic propositions?		
1   2 interpretations (rows)			
2 4 interpretations (rows)			
3 8 interpretations (rows)			E2.2 Equivale
n ??? interpretations			
Some examples: $2^{10} = 1024$ , $2^{20} = 1$ $\rightarrow$ not viable for larger formulas; we			
more on difficulty of satisfiability	vetc.		
Theory of Computer Science cou			
practical algorithms: Foundation			

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**Equivalent Formulas** 

Definition (Equivalence of Propositional Formulas) Two propositional formulas  $\varphi$  and  $\psi$  over A are (logically) equivalent ( $\varphi \equiv \psi$ ) if for all interpretations  $\mathcal{I}$  for A it is true that  $\mathcal{I} \models \varphi$  if and only if  $\mathcal{I} \models \psi$ .

German: logisch äquivalent

E2.2 Equivalences	

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E2. Properties of Formulas and Equivalences Equivalences Equivalent Formulas: Example  $((\varphi \lor \psi) \lor \chi) \equiv (\varphi \lor (\psi \lor \chi))$  $\mathcal{I} \models$  $\mathcal{I}\models \mathcal{I}\models$  $\mathcal{I} \models$  $\mathcal{I} \models$  $\mathcal{I} \models$  $\mathcal{I} \models$  $(\psi \stackrel{\cdot}{\vee} \chi) \quad ((\varphi \lor \dot{\psi}) \lor \chi) \quad (\varphi \lor (\dot{\psi} \lor \chi))$  $(\varphi \lor \psi)$  $\psi$  $\varphi$  $\chi$ No No No No No No No Yes No Yes No Yes Yes No Yes Yes Yes Yes No Yes No Yes Yes No Yes Yes Yes Yes No Yes Yes Yes No No Yes Yes No Yes Yes Yes Yes Yes Yes Yes No Yes Yes

Equivalences

Equivalences

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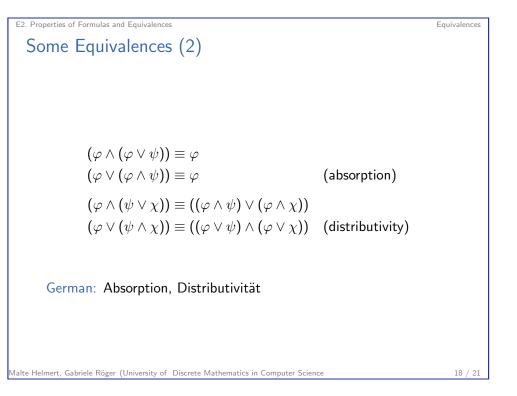
### Some Equivalences (1)

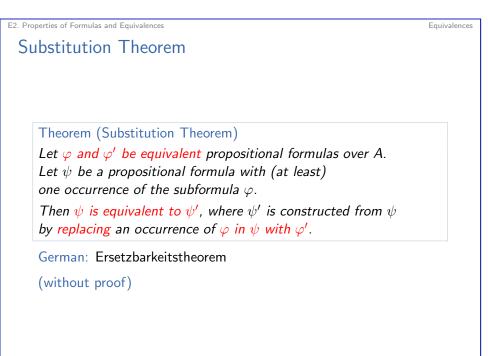
 $\begin{aligned} (\varphi \land \varphi) &\equiv \varphi \\ (\varphi \lor \varphi) &\equiv \varphi & \text{(idempotence)} \\ (\varphi \land \psi) &\equiv (\psi \land \varphi) \\ (\varphi \lor \psi) &\equiv (\psi \lor \varphi) & \text{(commutativity)} \\ ((\varphi \land \psi) \land \chi) &\equiv (\varphi \land (\psi \land \chi)) \\ ((\varphi \lor \psi) \lor \chi) &\equiv (\varphi \lor (\psi \lor \chi)) & \text{(associativity)} \end{aligned}$ 

German: Idempotenz, Kommutativität, Assoziativität

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E2. Properties of Formulas and Equivalences (3)  $\neg \neg \varphi \equiv \varphi \qquad (double negation) \\
\neg (\varphi \land \psi) \equiv (\neg \varphi \lor \neg \psi) \\
\neg (\varphi \lor \psi) \equiv (\neg \varphi \land \neg \psi) \qquad (De Morgan's rules) \\
(\varphi \lor \psi) \equiv \varphi \text{ if } \varphi \text{ tautology} \\
(\varphi \land \psi) \equiv \psi \text{ if } \varphi \text{ tautology} \qquad (tautology rules) \\
(\varphi \lor \psi) \equiv \psi \text{ if } \varphi \text{ unsatisfiable} \\
(\varphi \land \psi) \equiv \varphi \text{ if } \varphi \text{ unsatisfiable} \qquad (unsatisfiability rules)$ German: Doppelnegation, De Morgansche Regeln, Tautologieregeln, Unerfüllbarkeitsregeln





Equivalences

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