# Discrete Mathematics in Computer Science D2. Advanced Methods for Recurrences

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D2. Advanced Methods for Recurrences

Fibonacci Series – Generating Functions

1 / 30

# D2.1 Fibonacci Series – Generating Functions

Discrete Mathematics in Computer Science — D2. Advanced Methods for Recurrences

D2.1 Fibonacci Series – Generating Functions

D2.2 Master Theorem for Divide-and-Conquer Recurrences

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D2. Advanced Methods for Recurrences Revisiting the Fibonacci Series

- In this section we study generating functions,
  - a powerful method for solving recurrences.
- Generating functions allow us to directly derive closed-form expressions for recurrences.
- Full mastery of generating functions requires solid knowledge of calculus, in particular power series.
- Rather than develop the topic in its full depth, we will look at it within the context of a case study, proving the closed form of the Fibonacci series again.
- We leave out some of the more subtle mathematical aspects, such as the question of convergence of the power series used.

2 / 30

Fibonacci Series – Generating Functions

#### D2. Advanced Methods for Recurrences

Fibonacci Series – Generating Functions

## Power Series

Definition (power series) Let  $(a_n)_{n \in \mathbb{N}_0}$  be a sequence of real numbers. The power series with coefficients  $(a_n)$  is the (possibly partial) function  $g : \mathbb{R} \to \mathbb{R}$  defined by

 $g(x) = \sum_{n=0}^{\infty} a_n x^n$  for all  $x \in \mathbb{R}$ .

German: Potenzreihe

Notes: more general definitions exist, for example

- using  $(x c)^n$  instead of  $x^n$  for some  $c \in \mathbb{R}$
- using complex instead of real numbers
- using multiple variables

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D2. Advanced Methods for Recurrences **Uniqueness of Power Series Representation** Theorem Let g and h be power series with coefficients  $(a_n)$  and  $(b_n)$ . Let  $\varepsilon > 0$  such that for all  $|x| < \varepsilon$ :  $\bullet$  g and h converge, and  $\bullet$  g(x) = h(x). Then  $a_n = b_n$  for all  $n \in \mathbb{N}_0$ .

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Power Series – Examples

Reminder: g(x) = \sum_{n=0}^{\infty} a_n x^n

Examples:

• a_n = 1

\Rightarrow g(x) = \frac{1}{1-x} (only defined for |x| < 1)

• a_n = z^n for some z \in \mathbb{R}

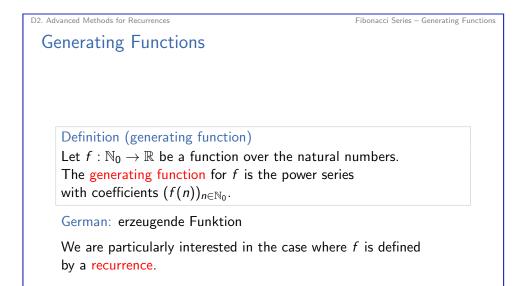
\Rightarrow g(x) = \frac{1}{1-zx} (only defined for |x| < 1/|z|)

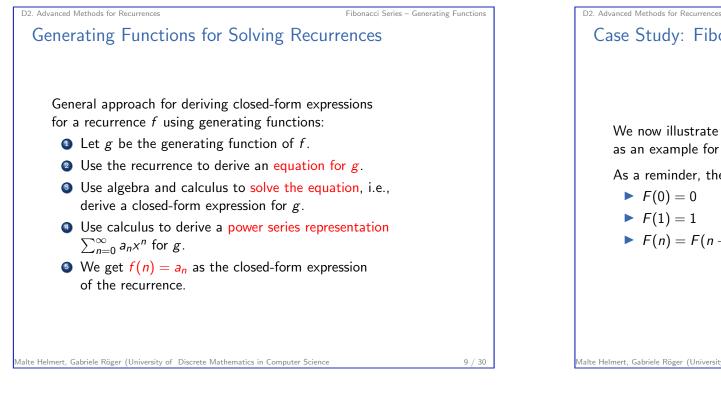
• a_n = \frac{1}{1-zx} (only defined for all x)

• a_n = \begin{cases} 0 & n \text{ is even} \\ \frac{(-1)^{(n-1)/2}}{n!} & x \text{ is odd} \\ \Rightarrow g(x) = \sin x \text{ (defined for all } x) \end{cases}
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6 / 30





Fibonacci Series – Generating Functions

Case Study: 1. Generating Function

1. Let g be the generating function of f.

$$g(x)=\sum_{n=0}^{\infty}F(n)x^n$$
 for  $x\in\mathbb{R}$ 

Note: The series does not converge for all x, but it converges for  $|x| < \varepsilon$  for sufficiently small  $\varepsilon > 0$ . We omit details.

# Case Study: Fibonacci Numbers

We now illustrate the approach using the Fibonacci numbers Fas an example for the recurrence f.

As a reminder, the Fibonacci numbers are defined as follows:

► F(n) = F(n-1) + F(n-2) for all n > 2

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D2. Advanced Methods for Recurrences Fibonacci Series – Generating Functions Case Study: 2. Equation for g from Recurrence F(1) = 1 F(n) = F(n-1) + F(n-2) for all  $n \ge 2$ F(0) = 02. Use the recurrence to derive an equation for g $g(x) = \sum_{n=0}^{\infty} F(n)x^n = 0 \cdot x^0 + 1 \cdot x^1 + \sum_{n=2}^{\infty} (F(n-1) + F(n-2))x^n$  $= x + \sum_{n=2}^{\infty} F(n-1)x^n + \sum_{n=2}^{\infty} F(n-2)x^n$  $= x + \sum_{n=1}^{\infty} F(n)x^{n+1} + \sum_{n=1}^{\infty} F(n)x^{n+2}$  $= x + x \sum_{n=1}^{\infty} F(n)x^n + x^2 \sum_{n=1}^{\infty} F(n)x^n$  $= x + x \sum_{n=0}^{\infty} F(n)x^n + x^2 \sum_{n=0}^{\infty} F(n)x^n$  $= x + x g(x) + x^2 g(x)$ 

#### D2. Advanced Methods for Recurrences

Fibonacci Series – Generating Functions

# Case Study: 3. Solve Equation for g

3. Use algebra and calculus to solve the equation, i.e., derive a closed-form expression for *g*.

$$g(x) = x + x g(x) + x^{2}g(x)$$

$$\Rightarrow \quad g(x) - x g(x) - x^{2}g(x) = x$$

$$\Rightarrow \quad g(x)(1 - x - x^{2}) = x$$

$$\Rightarrow \quad g(x) = \frac{x}{1 - x - x^{2}}$$

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D2. Advanced Methods for Recurrences Case Study: 4. Power Series Representation for g (2) 4. Use calculus to derive a power series representation  $\sum_{n=0}^{\infty} a_n x^n$  for g. (1) a + b = 1, (2)  $-a\beta - b\alpha = 0$ , (3)  $-\alpha - \beta = -1$ , (4)  $\alpha\beta = -1$ From (3): (5)  $\beta = 1 - \alpha$ Substituting (5) into (4):  $\alpha(1 - \alpha) = -1$   $\Rightarrow \alpha - \alpha^2 = -1$   $\Rightarrow \alpha^2 - \alpha - 1 = 0$   $\Rightarrow \alpha = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} \pm \sqrt{\frac{5}{4}}$   $\Rightarrow \alpha = \frac{1 \pm \sqrt{5}}{2}$  $\Rightarrow$  The solutions are  $\alpha = \varphi$  or  $\alpha = \psi$  from the previous chapter. Continue with (6)  $\alpha = \varphi$ .

13 / 30

#### D2. Advanced Methods for Recurrences

# Case Study: 4. Power Series Representation for g(1)

4. Use calculus to derive a power series representation  $\sum_{n=0}^{\infty} a_n x^n$  for g.

$$g(x) = \frac{x}{1-x-x^2} = xh(x)$$
 with  $h(x) = \frac{1}{1-x-x^2}$   
Idea: partial fraction decomposition, i.e.,  
find  $a, b, \alpha, \beta$  such that  $h(x) = \frac{a}{1-\alpha x} + \frac{b}{1-\beta x}$ .

$$\frac{a}{1-\alpha x} + \frac{b}{1-\beta x} = \frac{a(1-\beta x) + b(1-\alpha x)}{(1-\alpha x)(1-\beta x)}$$
$$= \frac{a-a\beta x + b - b\alpha x}{1-\alpha x - \beta x + \alpha \beta x^2}$$
$$= \frac{(a+b) + (-a\beta - b\alpha)x}{1+(-\alpha - \beta)x + \alpha \beta x^2}$$
$$\Rightarrow a+b=1, \quad -a\beta - b\alpha = 0, \quad -\alpha - \beta = -1, \quad \alpha\beta = -1$$

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**Case Study:** 4. Power Series Representation for g (3)  
4. Use calculus to derive a power series representation 
$$\sum_{n=0}^{\infty} a_n x^n$$
 for g.  
(1)  $a + b = 1$ , (2)  $-a\beta - b\alpha = 0$ , (3)  $-\alpha - \beta = -1$ , (4)  $\alpha\beta = -1$ ,  
(5)  $\beta = 1 - \alpha$ , (6)  $\alpha = \varphi$   
• Substituting (6) into (5): (7)  $\beta = 1 - \alpha = 1 - \varphi = \psi$ .  
• From (1): (8)  $b = 1 - a$   
• Substituting (6), (7), (8) into (2):  
 $-a(1 - \varphi) - (1 - a)\varphi = 0$   
 $\Rightarrow -a + a\varphi - \varphi + a\varphi = 0$   
 $\Rightarrow a(2\varphi - 1) = \varphi$   
 $\Rightarrow a = \frac{\varphi}{2\varphi - 1} = \frac{\varphi}{2 \cdot \frac{1}{2}(1 + \sqrt{5}) - 1} = \frac{1}{\sqrt{5}}\varphi$ 

# Case Study: 4. Power Series Representation for g(4)

4. Use calculus to derive a power series representation  $\sum_{n=0}^{\infty} a_n x^n$  for g. (8) b = 1 - a, (9)  $a = \frac{1}{\sqrt{5}}\varphi$ 

Substituting (9) into (8):

$$b = 1 - a$$

$$= 1 - \frac{1}{\sqrt{5}}\varphi$$

$$= \frac{\sqrt{5}}{\sqrt{5}} - \frac{\frac{1}{2}(1 + \sqrt{5})}{\sqrt{5}}$$

$$= -\frac{1}{\sqrt{5}}(-\sqrt{5} + \frac{1}{2} + \frac{1}{2}\sqrt{5})$$

$$= -\frac{1}{\sqrt{5}}(\frac{1}{2} - \frac{1}{2}\sqrt{5})$$

$$= -\frac{1}{\sqrt{5}}\psi$$
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Case Study: 5. Extract Closed Form of Recurrence

Use calculus to derive a power series representation ∑<sub>n=0</sub><sup>∞</sup> a<sub>n</sub>x<sup>n</sup> for g.
 We get f(n) = a<sub>n</sub> as the closed-form expression of the recurrence.

From

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} (\varphi^n - \psi^n) x^n$$

we conclude:

$$F(n) = rac{1}{\sqrt{5}}(arphi^n - \psi^n) ext{ for all } n \in \mathbb{N}_0$$

D2. Advanced Methods for Recurrences

#### Fibonacci Series – Generating Functions

# Case Study: 4. Power Series Representation for g(5)

4. Use calculus to derive a power series representation  $\sum_{n=0}^{\infty} a_n x^n$  for g.  $g(x) = xh(x), \quad h(x) = \frac{a}{1-\alpha x} + \frac{b}{1-\beta x},$  $\alpha = \varphi, \quad \beta = \psi, \quad a = \frac{1}{\sqrt{5}}\varphi, \quad b = -\frac{1}{\sqrt{5}}\psi$ 

#### Plugging everything in:

$$g(x) = x \left(\frac{1}{\sqrt{5}}\varphi \frac{1}{1-\varphi x} - \frac{1}{\sqrt{5}}\psi \frac{1}{1-\psi x}\right) = \frac{x}{\sqrt{5}} \left(\varphi \frac{1}{1-\varphi x} - \psi \frac{1}{1-\psi x}\right)$$
$$= \frac{x}{\sqrt{5}} \left(\varphi \sum_{n=0}^{\infty} \varphi^n x^n - \psi \sum_{n=0}^{\infty} \psi^n x^n\right)$$
$$= \frac{1}{\sqrt{5}} \left(\sum_{n=0}^{\infty} \varphi^{n+1} x^{n+1} - \sum_{n=0}^{\infty} \psi^{n+1} x^{n+1}\right)$$
$$= \frac{1}{\sqrt{5}} \left(\sum_{n=1}^{\infty} \varphi^n x^n - \sum_{n=1}^{\infty} \psi^n x^n\right) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{5}} (\varphi^n - \psi^n) x^n$$
$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} (\varphi^n - \psi^n) x^n$$
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# D2.2 Master Theorem for Divide-and-Conquer Recurrences

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Master Theorem for Divide-and-Conquer Recurrences

## Asymptotic Growth

- Run-time analysis usually focuses on the asymptotic growth rate of run-time.
- For example, we say "run-time grows at most quadratically" rather than saying that run-time for inputs of size n is 3n<sup>2</sup> + 17n + 8.

#### advantages:

- much simpler to study
- can abstract from minor implementation details

Master Theorem for Divide-and-Conquer Recurrences

# Divide-and-Conquer Algorithms

Recurrences frequently arise in the run-time analysis of divide-and-conquer algorithms.

#### ► Examples:

- Mergesort: sort a sequence by recursively sorting two smaller sequences, then merging them
- Binary search: find an element in a sorted sequence by identifying which half of the sequence must contain the element, then recursively searching it
- Quickselect: find the k-th smallest element in a sequence by recursive partitioning

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22 / 30

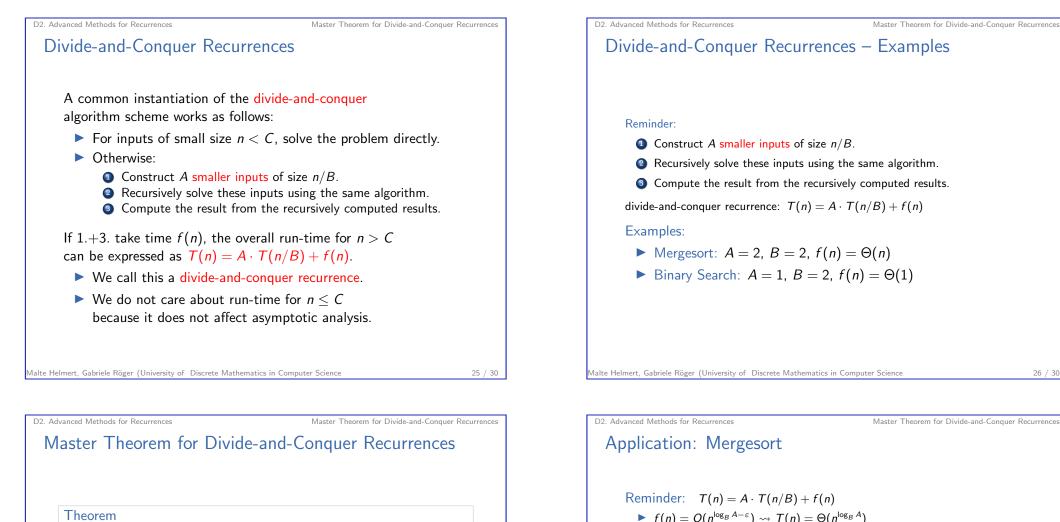
## D2. Advanced Methods for Recurrences Big-O, Big- $\Omega$ , Big- $\Theta$

### Definition $(O, \Omega, \Theta)$

Let  $g : \mathbb{R}_0^+ \to \mathbb{R}$  be a function. The sets of functions  $O(g), \Omega(g), \Theta(g)$  are defined as follows:  $\bullet O(g) = \{f : \mathbb{R}_0^+ \to \mathbb{R} \mid \text{there exist } C, n_0 \in \mathbb{R}$ s.t.  $|f(n)| \le C \cdot g(n) \text{ for all } n \ge n_0\}$   $\bullet \Omega(g) = \{f : \mathbb{R}_0^+ \to \mathbb{R} \mid \text{there exist } C, n_0 \in \mathbb{R}$ s.t.  $|f(n)| \ge C \cdot g(n) \text{ for all } n \ge n_0\}$  $\bullet \Theta(g) = O(g) \cap \Omega(g)$ 

#### Notation:

- ▶ It is convention to say " $5n^2 + 7n \log_2 n = \Theta(n^2)$ " instead of " $f \in \Theta(g)$  for the functions f, gwith  $f(n) = 5n^2 + 7n \log_2 n$  and  $g(n) = n^2$ ".
- ditto for O, Ω



Theorem

Let  $A \ge 1, B \ge 1$ , and let T satisfy the divide-and-conquer recurrence  $T(n) = A \cdot T(n/B) + f(n)$ . Then:

- If  $f(n) = O(n^{\log_B A \varepsilon})$  for some  $\varepsilon > 0$ . then  $T(n) = \Theta(n^{\log_B A})$ .
- $\blacktriangleright \quad If f(n) = \Theta(n^{\log_B A}).$ then  $T(n) = \Theta(n^{\log_B A} \log_2 n)$ .
- If  $f(n) = \Omega(n^{\log_B A + \varepsilon})$  for some  $\varepsilon > 0$ , then  $T(n) = \Theta(f(n))$ .

We do not prove the theorem.

•  $f(n) = \Theta(n^{\log_B A}) \rightsquigarrow T(n) = \Theta(n^{\log_B A} \log_2 n)$ 

 $\blacktriangleright f(n) = \Omega(n^{\log_B A + \varepsilon}) \rightsquigarrow T(n) = \Theta(f(n))$ 

Mergesort: A = 2, B = 2,  $f(n) = \Theta(n)$ 

 $\blacktriangleright f(n) = O(n^{1-\varepsilon}) \rightsquigarrow T(n) = \Theta(n^1)$ 

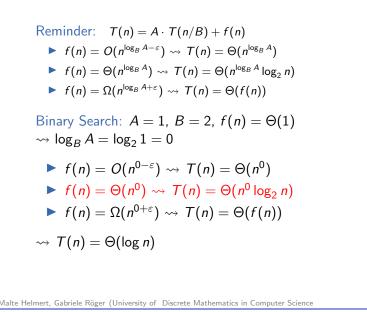
 $f(n) = \Theta(n^1) \rightsquigarrow T(n) = \Theta(n^1 \log_2 n)$ 

 $\blacktriangleright f(n) = \Omega(n^{1+\varepsilon}) \rightsquigarrow T(n) = \Theta(f(n))$ 

 $\rightsquigarrow \log_B A = \log_2 2 = 1$ 

 $\rightsquigarrow T(n) = \Theta(n \log n)$ 

29 / 30



D2. Advanced Methods for Recurrences

# More Complex Cases

Some divide-and-conquer algorithms have more complicated recurrences because they do not split into even-sized pieces of predictable size.

Example:

- Quicksort with random pivotization: f(n) = Θ(n); split n uniformly randomly into 1 ≤ k ≤ n and n − 1 − k ~ expected runtime Θ(n log n)
- Quickselect with median-of-median pivotization: f(n) = Θ(n); one recursion on input size n/5, one recursion on input size at most n ⋅ <sup>7</sup>/<sub>10</sub> → runtime Θ(n)

Here, we can try to use the Master theorem to derive hypotheses and then prove them by mathematical induction.

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