Discrete Mathematics in Computer Science

D1. Introduction to Recurrences

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D1.1 Recurrences

Recursion (1)

The concept of recursion is very common in computer science and discrete mathematics.

- When designing algorithms, recursion relates to the idea of solving a problem by solving smaller subproblems of the same kind.
- Examples:
  - For example, we can sort a sequence by sorting smaller subsequences and then combining the result ⇝ mergesort
  - We can find an element in a sorted sequence by identifying which half of the sequence the element must be located in, and then searching this half ⇝ binary search
  - We can insert elements into a search tree by identifying which child of the root node the element must be added to, then recursively inserting it there ⇝ trees as data structures

Recursion (2)

The concept of recursion is very common in computer science and discrete mathematics.

- When designing data structures, it is often helpful to think of a data structures as being composed of smaller data structures of the same kind.
- Examples:
  - A rooted binary tree is either a leaf or an inner node with two children, which are themselves rooted binary trees.
  - A singly linked list is either empty or a head element followed by a tail, which is itself a linked list.
  - A logical formula is either an atomic formula or a composite formula, which consists of one or two formulas connected by logical connectives ("and", "or", "not").

Recursion (3)

The concept of recursion is very common in computer science and discrete mathematics.

- In combinatorial counting problems, counting things often involves solving smaller counting problems of the same type and combining the results.
- Examples:
  - counting the number of subsets of size k of a set of size n
  - counting the number of permutations of a set of size n
  - counting the number of rooted binary trees with n leaves

Recurrences

In this part of the lecture, we study recurrences, i.e., recursively defined functions $f : \mathbb{N}_0 \to \mathbb{R}$ where $f(n)$ is defined in terms of the values $f(m)$ for $m < n$.

- Such recurrences naturally arise in all mentioned applications.
- They are particularly useful for studying the runtime of algorithms, especially recursive algorithms.

Learning Objectives

- Recurrences are a wide topic, and in our brief coverage we will only scratch the surface.
- Our aim is to equip you with enough knowledge to
  - understand what recurrences are
  - understand where they arise
  - understand why they are of interest
  - get to know some important examples of recurrences, such as the Fibonacci series
  - get a feeling for some mathematical techniques used to solve recurrences, in particular:
    - mathematical induction
    - generating functions
    - the master theorem for divide-and-conquer recurrences
  - apply the master theorem in practice

D1.2 Examples of Recurrences

Examples of Recurrences

In this section, we look at three recurrences that arise in combinatorics, i.e., when counting things:

- factorials: counting permutations
- binomial coefficients: counting subsets of a certain size
- Catalan numbers: counting rooted binary trees

We also have a first look at the Fibonacci series, perhaps the most famous recurrence in mathematics.

Counting Permutations

Let S be a finite set, and let $n = |S|$.

Question: How many permutations of S exist?

We answer this question by answering the following slightly more general question:

Let X and Y be finite sets, and let $n = |X| = |Y|$.

Question: How many bijective functions from X to Y exist?

The permutation question is the special case where $S = X = Y$.

Counting Bijections – Derivation

How many bijective functions from X to Y exist, where $n = |X| = |Y|$?

Denote this number by $f(n)$.

- We have $f(0) = 1$: there exists one possible function from $X = \emptyset$ to $Y = \emptyset$ (the empty function), and it is bijective.
- For $n \geq 1$, let $x \in X$ be any element of X.
  - Every bijection $g : X \to Y$ maps x to some element $g(x) = y \in Y$.
  - There are $n = |Y|$ possible choices for y.
- In order to be bijective, g must bijectively map all other elements in X to other elements of Y.
  - Hence, g restricted to $X \setminus \{x\}$ is a bijective function from $X \setminus \{x\}$ to $Y \setminus \{y\}$.
  - Because $|X \setminus \{x\}| = |Y \setminus \{y\}| = n - 1$, there are $f(n - 1)$ choices for these mappings.
- This gives us $f(n) = n \cdot f(n - 1)$ for all $n \geq 1$.

Counting Bijections – Result

Theorem

The number of bijections between finite sets of size n, or equivalently the number of permutations of a finite set of size n, is given by the recurrence:

$$f(0) = 1$$
$$f(n) = n \cdot f(n-1) \qquad \text{for all } n \geq 1$$

Closed-form solution:

$$f(n) = n!$$

Counting k-Subsets

Let S be a finite set, let $n = |S|$, and let $k \in \{0, \dots, n\}$.

Question: How many subsets of S of size k exist?

Denote this number by $\binom{n}{k}$.

- We have $\binom{n}{0} = 1$: the only subset of size 0 is $\emptyset$.
- We have $\binom{n}{n} = 1$: the only subset of size n is S itself.
- For all other cases, we count proper, nontrivial subsets. Let $x \in S$ be any element.
- There are two kinds of subsets of S of size k:
  - subsets that do not include x:
    Such subsets include k elements of the set $S \setminus \{x\}$.
    Because $|S \setminus \{x\}| = n-1$, there are $\binom{n-1}{k}$ such subsets.
  - subsets that include x:
    Such subsets include $k-1$ elements of $S \setminus \{x\}$.
    Because $|S \setminus \{x\}| = n-1$, there are $\binom{n-1}{k-1}$ such subsets.
- In summary: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for all $n \geq 1$ and $0 < k < n$.

Counting k-Subsets – Result

Theorem

Let S be a finite set with n elements, and let $k \in \{0, \dots, n\}$. Then S has $\binom{n}{k}$ subsets of size k, where

$$\binom{n}{0} = 1$$
$$\binom{n}{n} = 1$$
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad \text{for all } n \geq 1, 0 < k < n$$

Closed-form solution:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Counting k-Subsets – Proof of Closed-Form Solution

To prove that the given closed-form solution is correct, it suffices to verify that it satisfies the recurrence:

- case $k = 0$: $\frac{n!}{k!(n-k)!} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1 = \binom{n}{0}$.
- case $k = n$: $\frac{n!}{k!(n-k)!} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 0!} = \frac{n!}{n! \cdot 1} = 1 = \binom{n}{n}$.
- case $0 < k < n$:

$$\begin{aligned}
&\frac{(n-1)!}{k!((n-1)-k)!} + \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \\
&= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)! \cdot (n-k)}{k!(n-k-1)! \cdot (n-k)} + \frac{(n-1)! \cdot k}{(k-1)! \cdot k \cdot (n-k)!} \\
&= \frac{(n-1)! \cdot (n-k)}{k!(n-k)!} + \frac{(n-1)! \cdot k}{k! \cdot (n-k)!} \\
&= \frac{(n-1)! \cdot ((n-k)+k)}{k!(n-k)!} = \frac{(n-1)! \cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!}
\end{aligned}$$

Binary Trees

Definition (binary tree)

A binary tree is inductively defined as a tuple of the following form:

- The empty tree () is a binary tree.
  Such a tree is called a leaf.
- If L and R are binary trees, then (L, R) is a binary tree.
  Such a tree is called an inner node
  with left child L and right child R.

German: Binärbaum

Note: With these kinds of trees, the order of children matters, i.e., (L, R) and (R, L) are different trees (unless $L = R$).

Counting Binary Trees

Question: How many binary trees with $n + 1$ leaves exist?
(Why $n + 1$?)

Denote this number by $C(n)$.

- We have $C(0) = 1$: () is the only tree with one leaf.
- For $n \geq 1$, the tree must be an inner node.
  Each child must have between 1 and n leaves.
  The number of leaves of the children must sum to $n + 1$.
- Hence, if the left child has $k + 1$ leaves, the right child has $(n + 1) - (k + 1) = n - k = (n - k - 1) + 1$ leaves.
- We obtain: $C(n) = \sum_{k=0}^{n-1} C(k)C(n - k - 1)$.

Counting Binary Trees – Result

Theorem

There are $C(n)$ binary trees with $n + 1$ leaves, where

$$C(0) = 1$$

$$C(n) = \sum_{k=0}^{n-1} C(k)C(n - k - 1) \qquad \text{for all } n \geq 1$$

Closed-form solution (without proof):

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

Catalan Numbers

The numbers $C(n)$ are called Catalan numbers
after 19th century Belgian mathematician Eugène Charles Catalan.

First terms of the Catalan sequence:
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, ...

Fibonacci Series

- The last recurrence we consider in this section is the famous Fibonacci series (or Fibonacci sequence).
- We directly introduce it with its definition as a recurrence rather than via an application.

Fibonacci Series – Definition

Definition (Fibonacci series)

The Fibonacci series F is defined as follows:

$$\begin{aligned} F(0) &= 0 \\ F(1) &= 1 \\ F(n) &= F(n-1) + F(n-2) \qquad \text{for all } n \geq 2 \end{aligned}$$

German: Fibonacci-Folge

First terms of the Fibonacci series:
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

Closed-form solution: $\rightsquigarrow$ next section

Fibonacci Series – Trivia

- The Fibonacci series is named after Leonardo of Pisa a.k.a. Fibonacci (son of Bonacci), who introduced it to Western Europe in the 13th century.
- It has been known outside Europe much earlier, dating back to the Indian mathematician Pingala (3rd century BCE).
- The series has many, many applications.
- There exist mathematical journals solely dedicated to it, the most famous one being "Fibonacci Quarterly".

D1.3 Fibonacci Series – Mathematical Induction

Overview

- In this section, we prove a closed-form expression for the Fibonacci series.
- We do this because the result itself is interesting (because of the many applications of the Fibonacci series), but also to practice proving closed-form expressions for recurrences by mathematical induction.
- In the next section, we describe a more advanced technique with which we cannot just prove the given expression but also derive it ourselves.

Golden Ratio

Definition (golden ratio)

The number

$$\varphi = \frac{1+\sqrt{5}}{2}$$

is called the golden ratio.

German: goldener Schnitt

- Numerically, $\varphi = 1.618034$ (approximately).
- The golden ratio is a famous mathematical constant because it naturally occurs in many contexts and because of its aesthetical properties.

Negative Inverse of the Golden Ratio

Definition (negative inverse of the golden ratio)

The

$$\psi = \frac{1-\sqrt{5}}{2}$$

is called the negative inverse of the golden ratio.

- Numerically, $\psi = -0.618034$ (approximately).
- The name for ψ derives from the fact that $\psi = -\frac{1}{\varphi}$. However, we do not need this property here, and therefore we do not prove it.

Fibonacci Series – Closed-Form Expression

Theorem

$$\begin{aligned} F(n) &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \\ &= \frac{1}{\sqrt{5}}(\varphi^n - \psi^n) \qquad \text{for all } n \geq 0 \end{aligned}$$

Before we prove the theorem, we prove a number of lemmas.

- Note that $|\psi| < 1$ and hence $\psi^n \to 0$ as $n \to \infty$.
- With some calculation, we see that we can alternatively compute $F(n)$ by rounding $\frac{1}{\sqrt{5}}\varphi^n$ to the nearest integer, ignoring the ψ^n term.

First Lemma

Lemma

$$\psi = 1 - \varphi$$

Proof.

$$\begin{aligned}\psi &= \frac{1-\sqrt{5}}{2} \\ &= \frac{1+1-1-\sqrt{5}}{2} \\ &= \frac{2-(1+\sqrt{5})}{2} \\ &= \frac{2}{2} - \frac{1+\sqrt{5}}{2} \\ &= 1 - \varphi\end{aligned}$$

□

Second Lemma

Lemma

$$\varphi^2 = \varphi + 1$$

Proof.

$$\begin{aligned}\varphi^2 &= \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1}{4}(1+\sqrt{5})^2 \\ &= \frac{1}{4}(1+2\sqrt{5}+5) \\ &= \frac{1}{4}(2+2\sqrt{5}+4) = \frac{1}{4}(2+2\sqrt{5}) + \frac{4}{4} \\ &= \frac{1}{2}(1+\sqrt{5}) + 1 \\ &= \varphi + 1\end{aligned}$$

□

Third Lemma

Lemma

$$\psi^2 = \psi + 1$$

Proof.

$$\begin{aligned}\psi^2 &= (1-\varphi)^2 \\ &= 1 - 2\varphi + \varphi^2 \\ &= 1 - 2\varphi + \varphi + 1 \\ &= 1 - \varphi + 1 \\ &= (1-\varphi) + 1 \\ &= \psi + 1\end{aligned}$$

□

Main Proof (1)

Reminders:

$F(0) = 0$ $F(1) = 1$ $F(n) = F(n-1) + F(n-2)$ for all $n \geq 2$

$\varphi^2 = \varphi + 1$ $\psi^2 = \psi + 1$ Claim: $F(n) = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)$

Proof.

Proof by (strong) induction over n.

First base case $n = 0$:

$\frac{1}{\sqrt{5}}(\varphi^0 - \psi^0) = \frac{1}{\sqrt{5}}(1-1) = 0 = F(0)$

Second base case $n = 1$:

$\frac{1}{\sqrt{5}}(\varphi^1 - \psi^1) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}-1+\sqrt{5}}{2})$

$= \frac{1}{\sqrt{5}}(\frac{2\sqrt{5}}{2}) = 1 = F(1)$ ...

Main Proof (2)

Reminders:

$F(0) = 0$ $F(1) = 1$ $F(n) = F(n-1) + F(n-2)$ for all $n \geq 2$

$\varphi^2 = \varphi + 1$ $\psi^2 = \psi + 1$ Claim: $F(n) = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)$

Proof (continued).

Induction step (n building on $n-1$ and $n-2$):

$$
\begin{aligned}
F(n) &= F(n-1) + F(n-2) \\
&= \frac{1}{\sqrt{5}}(\varphi^{n-1} - \psi^{n-1}) + \frac{1}{\sqrt{5}}(\varphi^{n-2} - \psi^{n-2}) \\
&= \frac{1}{\sqrt{5}}(\varphi^{n-1} + \varphi^{n-2} - (\psi^{n-1} + \psi^{n-2})) \\
&= \frac{1}{\sqrt{5}}(\varphi^{n-2}(\varphi + 1) - \psi^{n-2}(\psi + 1)) \\
&= \frac{1}{\sqrt{5}}(\varphi^{n-2} \cdot \varphi^2 - \psi^{n-2} \cdot \psi^2) = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)
\end{aligned}
$$

□