Discrete Mathematics in Computer Science

Relations

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Relations: Informally

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- We already know some relations, e. g.
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- These are examples of binary relations, considering pairs of objects.
- There are also relations of higher arity, e. g.
  - "$x + y = z$" for integers x, y, z.
  - "The name, address and office number belong to the same person."

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  - "$x + y = z$" for integers x, y, z.
  - "The name, address and office number belong to the same person."
- Relations are for example important for relational databases, semantic networks or knowledge representation and reasoning.

Relations

Definition (Relation)

Let $S_1, \ldots, S_n$ be sets.

A relation over $S_1, \ldots, S_n$ is a set $R \subseteq S_1 \times \cdots \times S_n$.

The arity of R is n.

- A relation of arity n is a set of n-tuples.
- The set contains the tuples
  for which the informal property is true.

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- $R' = \{(\text{Gabi}, \text{Spiegelgasse 1}, 04.005),$
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Properties of Binary Relations

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- Instead of $(x, y) \in R$, we also write xRy, e. g.
  $x \leq y$ instead of $(x, y) \in \ \leq$
- If the sets are equal, we say "R is a binary relation over A" instead of "R is a binary relation over A and A".
- Such a relation over a set is also called
  a homogeneous relation or an endorelation.

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How do these properties relate to irreflexivity?

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Special Classes of Relations

- Some important classes of relations are defined in terms of these properties.
  - Equivalence relation: reflexive, symmetric, transitive
  - Partial order: reflexive, antisymmetric, transitive
  - Strict order: irreflexive, asymmetric, transitive
  - ...
- We will consider these and other classes in detail.