Discrete Mathematics in Computer Science

Cantor's Theorem

Malte Helmert, Gabriele Röger

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Countable Sets

We already know:

- The cardinality of $\mathbb{N}_0$ is $\aleph_0$.
- All sets with cardinality $\aleph_0$ are called countably infinite.
- A countable set is finite or countably infinite.
- Every subset of a countable set is countable.
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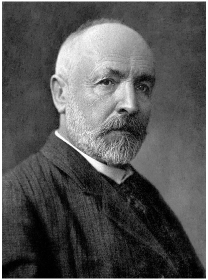
These questions were still open:

- Do all infinite sets have the same cardinality?
- Does the power set of infinite set S
  have the same cardinality as S?

Georg Cantor



- German mathematician (1845–1918)
- Proved that the rational numbers are countable.
- Proved that the real numbers are not countable.
- Cantor's Theorem: For every set S it holds that $|S| < |\mathcal{P}(S)|$.

Our Plan

- Understand Cantor's theorem
- Understand an important theoretical implication for computer science

Cantor's Diagonal Argument Illustrated on a Finite Set

$S = \{a, b, c\}$.

Consider an arbitrary injective function from S to $\mathcal{P}(S)$.
For example:

	a	b	c	
a	1	0	1	a mapped to $\{a, c\}$
b	1	1	0	b mapped to $\{a, b\}$
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$$\begin{array}{cccccl} & a & b & c & & \\ a & 1 & 0 & 1 & & a \text{ mapped to } \{a, c\} \\ b & 1 & 1 & 0 & & b \text{ mapped to } \{a, b\} \\ c & 0 & 1 & 0 & & c \text{ mapped to } \{b\} \end{array}$$

We can identify an "unused" element of $\mathcal{P}(S)$.

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Complement the entries on the main diagonal.

Cantor's Diagonal Argument Illustrated on a Finite Set

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Complement the entries on the main diagonal.

Works with every injective function from S to $\mathcal{P}(S)$.
$\rightarrow$ there cannot be a bijection from S to $\mathcal{P}(S)$.

Cantor's Diagonal Argument on a Countably Infinite Set

$S = \mathbb{N}_0$.

Consider an arbitrary injective function from $\mathbb{N}_0$ to $\mathcal{P}(\mathbb{N}_0)$.
For example:

$$
\begin{array}{ccccccc}
 & 0 & 1 & 2 & 3 & 4 & \dots \\
0 & 1 & 0 & 1 & 0 & 1 & \dots \\
1 & 1 & 1 & 0 & 1 & 0 & \dots \\
2 & 0 & 1 & 0 & 1 & 0 & \dots \\
3 & 1 & 1 & 0 & 0 & 0 & \dots \\
4 & 1 & 1 & 0 & 1 & 1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

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\begin{array}{c|cccccc}
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0 & \color{red}{1} & 0 & 1 & 0 & 1 & \dots \\
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4 & 1 & 1 & 0 & 1 & \color{red}{1} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
 & 0 & 0 & 1 & 1 & 0 & \dots
\end{array}
$$

Complementing the entries on the main diagonal
again results in an "unused" element of $\mathcal{P}(\mathbb{N}_0)$.

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We need to show that

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2. There is no bijection from S to $\mathcal{P}(S)$.

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For 1, consider function $f : S \to \mathcal{P}(S)$ with $f(x) = \{x\}$. Each element of S is paired with a unique element of $\mathcal{P}(S)$. . . .

Cantor's Theorem

Proof (continued).

For 2, we show for every injective function $f : S \to \mathcal{P}(S)$
that it is not a bijection from S to $\mathcal{P}(S)$.
This is sufficient because every bijection is injective.

Cantor's Theorem

Proof (continued).

For 2, we show for every injective function $f : S \to \mathcal{P}(S)$
that it is not a bijection from S to $\mathcal{P}(S)$.
This is sufficient because every bijection is injective.

Let f be an arbitrary injective function with $f : S \to \mathcal{P}(S)$.

Consider $M = \{x \mid x \in S, x \notin f(x)\}$.

For every $x \in S$ it holds that $f(x) \neq M$ because
$x \in f(x)$ iff not $x \notin f(x)$ iff not $x \in M$ iff $x \notin M$.

Hence, there is no $x \in S$ with $f(x) = M$. As $M \in \mathcal{P}(S)$ this implies that f is not a bijection from S to $\mathcal{P}(S)$. □

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Consequences of Cantor's Theorem

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Infinite Sets can Have Different Cardinalities

There are infinitely many different cardinalities of infinite sets:

- $|\mathbb{N}_0| < |\mathcal{P}(\mathbb{N}_0))| < |\mathcal{P}(\mathcal{P}(\mathbb{N}_0)))| < \ldots$
- $|\mathbb{N}_0| = \aleph_0 = \beth_0$
- $|\mathcal{P}(\mathbb{N}_0)| = \beth_1 (= |\mathbb{R}|)$
- $|\mathcal{P}(\mathcal{P}(\mathbb{N}_0))| = \beth_2$
- $\ldots$

Existence of Unsolvable Problems

There are more problems in computer science
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Why can we say so?

Decision Problems

"Intuitive Definition:" Decision Problem

A decision problem is a Yes-No question of the form
"Does the given input have a certain property?"

- "Does the given binary tree have more than three leaves?"
- "Is the given integer odd?"
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- A computer program solves a decision problem if it terminates on every input and returns the correct answer.

More Problems than Programs I

- A computer program is given by a finite string.
- A decision problem corresponds to a set of strings.

More Problems than Programs II

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  - every subset of S corresponds to a separate decision problem
- By Cantor's theorem $|S| < |\mathcal{P}(S)|$,
  so there are more problems than programs.

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Sets: Summary

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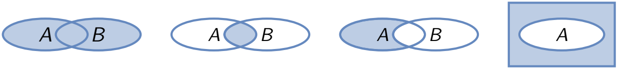
- Set operations: union, intersection, set difference, complement



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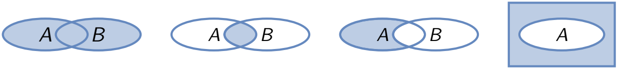

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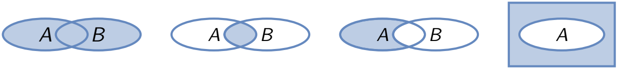

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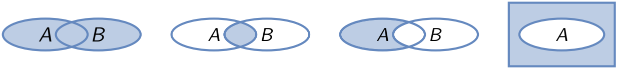

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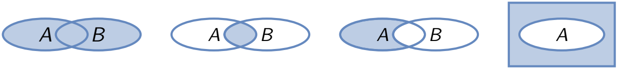

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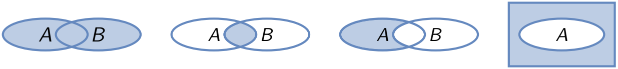

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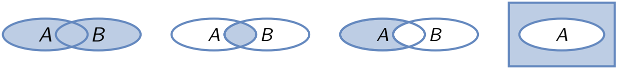

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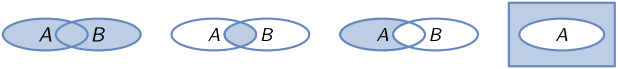

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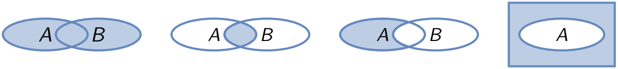

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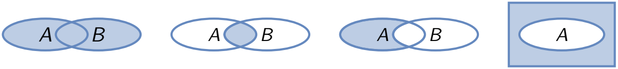

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  - For all sets S it holds that $|S| < |\mathcal{P}(S)|$.