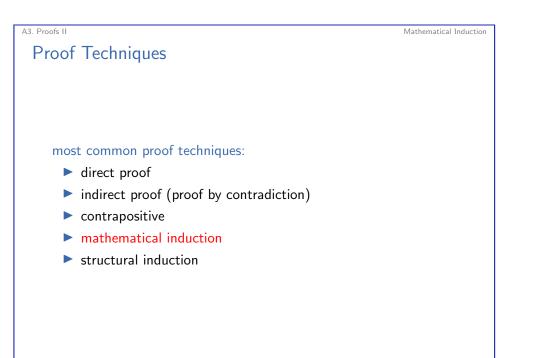


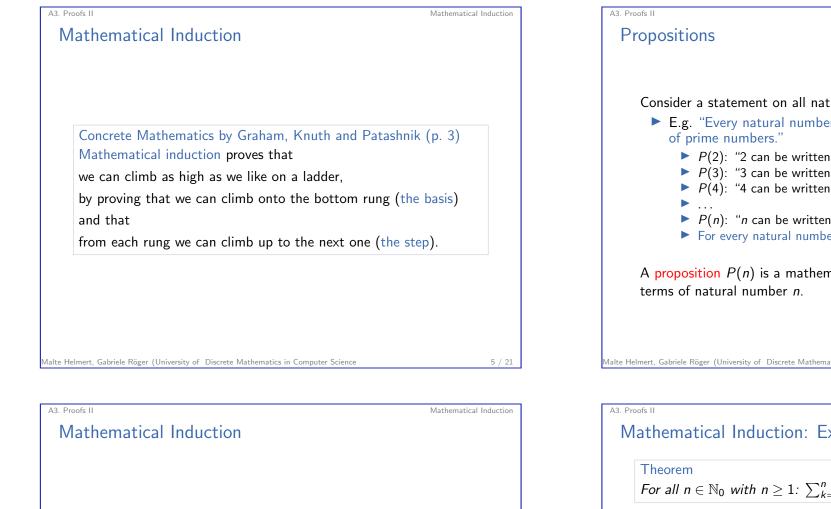
A3.1 Mathematical Induction

Discrete Mathematics in Computer Science – A3. Proofs II	
A3.1 Mathematical Induction	
A3.2 Structural Induction	

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Mathematical Induction

Proof (of the truth) of proposition P(n)for all natural numbers *n* with $n \ge m$:

- **basis**: proof of P(m)
- induction hypothesis (IH): suppose that P(k) is true for all k with $m \le k \le n$
- inductive step: proof of P(n+1)using the induction hypothesis

Mathematical Induction

Consider a statement on all natural numbers *n* with n > m.

- **•** E.g. "Every natural number $n \ge 2$ can be written as a product
 - \triangleright P(2): "2 can be written as a product of prime numbers."
 - \triangleright P(3): "3 can be written as a product of prime numbers."
 - \triangleright P(4): "4 can be written as a product of prime numbers."
 - \triangleright P(n): "*n* can be written as a product of prime numbers."
 - For every natural number n > 2 proposition P(n) is true.

A proposition P(n) is a mathematical statement that is defined in

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Mathematical Induction

Mathematical Induction: Example I

For all $n \in \mathbb{N}_0$ with $n \ge 1$: $\sum_{k=1}^n (2k-1) = n^2$

Proof.

Mathematical induction over *n*:

basis n = 1: $\sum_{k=1}^{1} (2k - 1) = 2 - 1 = 1 = 1^2$ IH: $\sum_{k=1}^{m} (2k-1) = m^2$ for all $1 \le m \le n$ inductive step $n \rightarrow n + 1$:

$$\sum_{k=1}^{n+1} (2k-1) = \left(\sum_{k=1}^{n} (2k-1)\right) + 2(n+1) - 1$$
$$\stackrel{\text{lH}}{=} n^2 + 2(n+1) - 1$$
$$= n^2 + 2n + 1 = (n+1)^2$$

A3. Proofs II

Mathematical Induction

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Mathematical Induction: Example II

Theorem

Every natural number $n \ge 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$ with prime numbers p_1, \ldots, p_m .

Proof.

Mathematical Induction over *n*:

- basis n = 2: trivially satisfied, since 2 is prime
- IH: Every natural number k with $2 \le k \le n$
 - can be written as a product of prime numbers.

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A3. Proofs II Weak vs. Strong Induction
Weak vs. Strong Induction
Weak induction: Induction hypothesis only supposes that P(k) is true for k = n
Strong induction: Induction hypothesis supposes that P(k) is true for all k ∈ N₀ with m ≤ k ≤ n
also: complete induction
Our previous definition corresponds to strong induction.
Which of the examples had also worked with weak induction?

Mathematical Induction: Example II

Theorem

Every natural number $n \ge 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$ with prime numbers p_1, \ldots, p_m .

Proof (continued).

inductive step $n \rightarrow n + 1$:

- **Case 1**: n + 1 is a prime number \rightsquigarrow trivial
- ► Case 2: n + 1 is not a prime number. There are natural numbers $2 \le q, r \le n$ with $n + 1 = q \cdot r$. Using IH shows that there are prime numbers q_1, \ldots, q_s with $q = q_1 \cdot \ldots \cdot q_s$ and r_1, \ldots, r_t with $r = r_1 \cdot \ldots \cdot r_t$. Together this means $n + 1 = q_1 \cdot \ldots \cdot q_s \cdot r_1 \cdot \ldots \cdot r_t$.

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Mathematical Induction

A3. Proofs II

Is Strong Induction More Powerful than Weak Induction?

Are there statements that we can prove with strong induction but not with weak induction?

We can always use a stronger proposition:

- "Every n ∈ N₀ with n ≥ 2 can be written as a product of prime numbers."
- \triangleright P(n): "*n* can be written as a product of prime numbers."
- ▶ P'(n): "all $k \in \mathbb{N}_0$ with $2 \le k \le n$ can be written as a product of prime numbers."



Reformulating Statements

It is sometimes convenient to rephrase a statement.

For example:

- "7" + 3" is divisible by 10 for all odd $n \in \mathbb{N}_0$."
- "For all $n \in \mathbb{N}_0$: if *n* is odd then $7^n + 3^n$ is divisible by 10."
 - P(n) = "if n is odd then $7^n + 3^n$ is divisible by 10."
 - Need two base cases.
 - Case distinction (n even or odd) in inductive step
- "For all $n \in \mathbb{N}_0$: $7^{(2n+1)} + 3^{(2n+1)}$ is divisible by 10."
 - $P'(n) = "7^{(2n+1)} + 3^{(2n+1)}$ is divisible by 10."

Be careful about how to reformulate a statement!

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Structural Induction

Mathematical Induction

A3. Proofs II Inductively Defined Sets: Examples

Example (Natural Numbers)

The set \mathbb{N}_0 of natural numbers is inductively defined as follows:

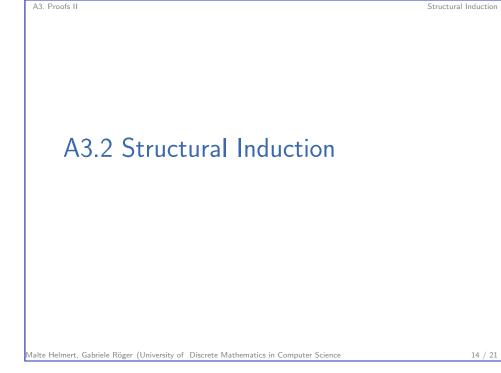
- ► 0 is a natural number.
- If *n* is a natural number, then n + 1 is a natural number.

Example (Binary Tree)

The set $\mathcal B$ of binary trees is inductively defined as follows:

- ▶ □ is a binary tree (a leaf)
- If L and R are binary trees, then ⟨L, ○, R⟩ is a binary tree (with inner node ○).

Implicit statement: all elements of the set can be constructed by finite application of these rules



A3. Proofs II Structural Induction Inductive Definition of a Set Inductive Definition A set *M* can be defined inductively by specifying basic elements that are contained in *M* construction rules of the form "Given some elements of *M*, another element of *M* can be constructed like this."

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Structural Induction

Structural Induction

Proof of statement for all elements of an inductively defined set

- basis: proof of the statement for the basic elements
- induction hypothesis (IH): suppose that the statement is true for some elements M
- inductive step: proof of the statement for elements constructed by applying a construction rule to M (one inductive step for each construction rule)

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Structural Induction

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Structural Induction: Example (2)

Theorem

A3. Proofs II

For all binary trees B: inner(B) = leaves(B) - 1.

Proof.

induction basis: $inner(\Box) = 0 = 1 - 1 = leaves(\Box) - 1$

 \rightsquigarrow statement is true for base case

A3. Proofs II

Structural Induction: Example (1)

Definition (Leaves of a Binary Tree)

The number of leaves of a binary tree B, written leaves(B), is defined as follows:

 $\begin{aligned} \textit{leaves}(\Box) &= 1\\ \textit{leaves}(\langle L, \bigcirc, R \rangle) &= \textit{leaves}(L) + \textit{leaves}(R) \end{aligned}$

Definition (Inner Nodes of a Binary Tree) The number of inner nodes of a binary tree *B*, written *inner*(*B*), is defined as follows:

$$inner(\Box) = 0$$

 $inner(\langle L, \bigcirc, R \rangle) = inner(L) + inner(R) + 1$

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Structural Induction

Structural Induction

Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree (L, \bigcirc, R) , we may use that it is true for the subtrees L and R.

inductive step for $B = \langle L, \bigcirc, R \rangle$:

$$inner(B) = inner(L) + inner(R) + 1$$

$$\stackrel{\text{IH}}{=} (leaves(L) - 1) + (leaves(R) - 1) + 1$$

$$= leaves(L) + leaves(R) - 1 = leaves(B) - 1$$

