Examples for Proof Techniques I

Gabi Röger

1 Basic Definitions

We will use sets to illustrate some proof techniques. For this purpose, we start with a few basic definitions.

Definition 1. *A* set *is an unordered collection of distinct objects.*

The objects in a set are called the elements *of the set. A set is said to* contain *its elements.*

We write $x \in S$ to indicate that x is an element of set S, and $x \notin S$ to indicate that S does not contain x.

The set that does not contain any elements is the empty set $\emptyset$.

We can express sets in terms of other sets:

Definition 2. *For sets A and B,*

- *the* union $A \cup B$ *is the smallest set that contains all elements of A and all elements of B;*
- *the* intersection $A \cap B$ *is the smallest set that contains all elements of A that are elements of B;*
- *the* set difference $A \setminus B$ *is the smallest set that contains all elements of A that are* not *elements of B.*

We also can compare sets. Two interesting properties are equality and the subset relation:

Definition 3. *Two sets A and B are* equal *(written $A = B$) if they contain exactly the same elements.*

We say that set A is a subset *of set B (written $A \subseteq B$) if every element of A is an element of B.*

2 Example: Direct Proof

We will use a direct proof to establish the following theorem:

Theorem 1. *For all sets A, B and C it holds that*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof. Let A, B and C be arbitrary sets. We will show separately that

- $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$ and that
- $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

We first show that $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$:

Consider any $x \in A \cap (B \cup C)$. By the definition of $\cap$ it holds that $x \in A$ and that $x \in (B \cup C)$. We make a case distinction between $x \in B$ and $x \notin B$:

Case 1 ($x \in B$): As $x \in A$ is true, it holds in this case that $x \in (A \cap B)$.

Case 2 ($x \notin B$): From $x \in (B \cup C)$ it follows for this case that $x \in C$. This implies together with $x \in A$ that $x \in (A \cap C)$.

In both cases it holds that $x \in A \cap B$ or $x \in A \cap C$, and we conclude that $x \in (A \cap B) \cup (A \cap C)$.

As x was chosen arbitrarily from $A \cap (B \cup C)$ we have shown that every element of $A \cap (B \cup C)$ is an element of $(A \cap B) \cup (A \cap C)$.

We will now show that every element of $(A \cap B) \cup (A \cap C)$ is an element of $A \cap (B \cup C)$.

... **[Homework assignment]** ...

Overall we have shown for arbitrary sets A, B and C that $x \in A \cap (B \cup C)$ if and only if $x \in (A \cap B) \cup (A \cap C)$, which concludes the proof of the theorem. □

3 Example: Proof by Contradiction

We will use a proof by contradiction to establish the following theorem:

Theorem 2. *For any sets A and B: If $A \setminus B = \emptyset$ then $A \subseteq B$.*

Proof. Assume that there are sets A and B with $A \setminus B = \emptyset$ and $A \not\subseteq B$.

Let A and B be such sets. Since $A \not\subseteq B$ there is some $x \in A$ such that $x \notin B$. For this x it holds that $x \in A \setminus B$. This is a contradiction to $A \setminus B = \emptyset$. □

4 Example: Proof by Contrapositive

We will prove the following theorem by contrapositive:

Theorem 3. *For any sets A and B: If $A \subseteq B$ then $A \setminus B = \emptyset$.*

Proof. We prove the theorem by contrapositive, showing for any sets A and B that if $A \setminus B \neq \emptyset$ then $A \not\subseteq B$.

Let A and B be arbitrary sets with $A \setminus B \neq \emptyset$. As the set difference is not empty, there is at least one x with $x \in A \setminus B$. By the definition of the set difference ($\setminus$), it holds that $x \in A$ and $x \notin B$. Hence, not all elements of A are elements of B, so it does not hold that $A \subseteq B$. □