Planning and Optimization
F3. Policy Iteration

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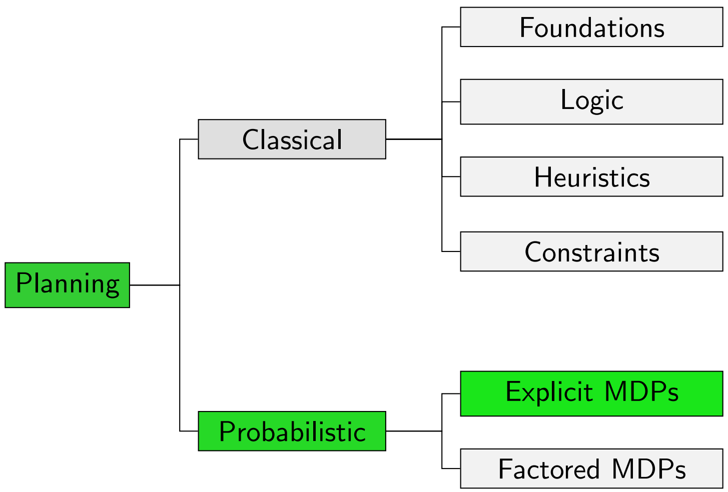
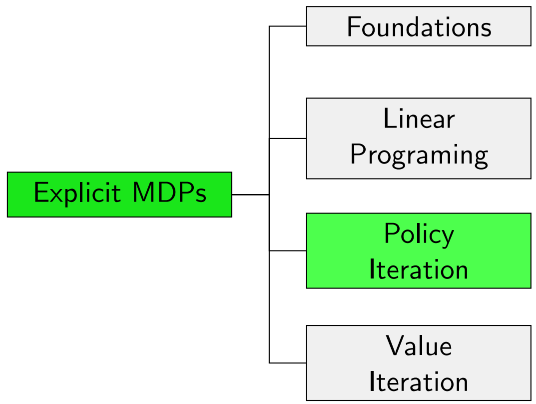
Content of this Course



Content of this Course: Explicit MDPs



F3.1 Introduction

Limitations of LPs in Practice

With the LP we can compute an optimal policy in polynomial time.

Possible issues in practice:

- LPs often too expensive even for small MDPs
- LP solver usage prohibited
- More expressive model required (e.g. continuous state space)

Policy Iteration (PI) is a suitable alternative. It has 2 components:

- Policy Evaluation: Compute V_π for a given π
- Policy Improvement: Determine better policy from V_π

F3.2 Policy Evaluation

Reminder: Value Functions for SSPs

Definition (Value Functions for SSPs)

Let $\mathcal{T} = \langle S, A, c, T, s_0, S_\star \rangle$ be an SSP and π be a policy for $\mathcal{T}$. The state-value $V_\pi(s)$ of s under π is defined as

$$V_\pi(s) := \begin{cases} 0 & \text{if } s \in S_\star \\ Q_\pi(s, \pi(s)) & \text{otherwise,} \end{cases}$$

where the action-value $Q_\pi(s, a)$ of s and a under π is defined as

$$Q_\pi(s, a) := c(a) + \sum_{s' \in \text{succ}(s,a)} T(s, a, s') \cdot V_\pi(s').$$

The state-value $V_\pi(s)$ describes the expected cost of applying π in SSP $\mathcal{T}$, starting from s.

Policy Evaluation: Implementations

Computing V_π for a given policy π is called policy evaluation.

There are several algorithms for policy evaluation:

1. Linear Program

Reminder: LP for Expected Cost in SSP

Variables

Non-negative variable ExpCost_s for each state s

Objective

Maximize ExpCost_{s_0}

Subject to

$$\text{ExpCost}_{s_\star} = 0 \quad \text{for all goal states } s_\star$$

$$\text{ExpCost}_s \leq \left(\sum_{s' \in S} T(s,a,s') \cdot \text{ExpCost}_{s'}\right) + c(a) \quad \text{for all } s \in S \text{ and } a \in A(s)$$

LP for Policy Evaluation in SSP

Variables

Non-negative variable ExpCost_s for each state s

Objective

Maximize ExpCost_{s_0}

Subject to

$$\text{ExpCost}_{s_\star} = 0 \quad \text{for all goal states } s_\star$$

$$\text{ExpCost}_s \leq \left(\sum_{s' \in S} T(s,\pi(s),s') \cdot \text{ExpCost}_{s'}\right) + c(\pi(s)) \quad \text{for all } s \in S \text{ ~~and } a \in A(s)\text{~~}$$

Policy Evaluation via LP

- is polynomial in $|S|$
- difference between polynomial in $|S|$ and polynomial in $|S| \cdot |A|$ is sometimes relevant in practice
- but often this is not the case
- other practical limitations also apply here

⇝ Require policy evaluation without LP

Policy Evaluation: Implementations

Computing V_π for a given policy π is called policy evaluation.

There are several algorithms for policy evaluation:

1. Linear Program
2. Backward Induction

Example: Backward Induction in Deterministic SSP

	1	2	3	4
5	⇒ 5.00	⇒ 2.00	⇒ 1.00	$s_\star$ 0.00
4	⇒ 6.00	⇑ 3.00	⇑ 4.00	⇑ 3.00
3	⇒ 7.00	⇑ 4.00	⇐ 5.00	⇐ 8.00
2	⇑ 10.00	⇑ 7.00	⇑ 6.00	⇐ 9.00
1	⇒ s_0 9.00	⇒ 8.00	⇑ 7.00	⇐ 10.00

- cost of 3 to move from striped cells (cost is 1 otherwise)

Policy Evaluation via Backward Induction

- is linear in $|S|$
- but restricted to special cases

⇝ When is policy evaluation via backward induction possible?

In deterministic planning problems?

Example: Backward Induction in Probabilistic SSP

	1	2	3	4
5	⇒ 5.00	⇒ 2.00	⇒ 1.00	$s_\star$ 0.00
4	⇒ 6.00	⇑ 3.00	⇑ 2.80	⇑ 3.00
3	⇒ 7.00	⇑ 4.00	⇐ 5.00	⇐ 8.00
2	⇑ 10.00	⇑ 7.00	⇑ 6.00	⇐ 9.00
1	⇒ s_0 9.00	⇒ 8.00	⇑ 7.00	⇐ 10.00

- cost of 3 to move from striped cells (cost is 1 otherwise)
- probability of 0.4 to "⇒" in gray cell

Policy Evaluation via Backward Induction

⇝ When is policy evaluation via backward induction possible?

In deterministic planning problems?
No, policy must be acyclic.

Policy Evaluation: Implementations

Computing V_π for a given policy π is called policy evaluation.

There are several algorithms for policy evaluation:

1. Linear Program
2. Backward Induction for acyclic policies

Backward Induction: Algorithm

Backward Induction for SSP $\langle S, A, c, T, s_0, S_\star \rangle$
and complete policy π

initialize $V_\pi(s) :=$ none for all $s \in S$
$V_\pi(s) := 0$ for all $s \in S_\star$
while there is a $s \in S$ with $V_\pi(s) =$ none:
  pick $s \in S$ with $V_\pi(s) =$ none and
    $V_\pi(s') \neq$ none for all $s' \in \text{succ}(s, \pi(s))$
  set $V_\pi(s) := c(\pi(s)) + \sum_{s' \in S} T(s, \pi(s), s') \cdot V_\pi(s')$
return V_π

Policy Evaluation: Implementations

Computing V_π for a given policy π is called policy evaluation.

There are several algorithms for policy evaluation:

1. Linear Program
2. Backward Induction for acyclic policies
3. Iterative Policy Evaluation

Iterative Policy Evaluation: Idea

- impossible to compute state-values in one sweep over the state space in presence of cycles
- start with arbitrary state-value function $\hat{V}_\pi^0$
- treat state-value function as update rule

$$\hat{V}_\pi^i(s) = c(\pi(s)) + \sum_{s' \in S} T(s, \pi(s), s') \cdot \hat{V}_\pi^{i-1}(s')$$

- apply update rule iteratively
- until state-values have converged

Iterative Policy Evaluation for SSPs: Example

	1	2	3	4
5	$\Rightarrow$ 0.00	$\Rightarrow$ 0.00	$\Rightarrow$ 0.00	$s_\star$ 0.00
4	$\Rightarrow$ 0.00	$\Uparrow$ 0.00	$\Uparrow$ 0.00	$\Uparrow$ 0.00
3	$\Rightarrow$ 0.00	$\Uparrow$ 0.00	$\Leftarrow$ 0.00	$\Leftarrow$ 0.00
2	$\Uparrow$ 0.00	$\Uparrow$ 0.00	$\Uparrow$ 0.00	$\Leftarrow$ 0.00
1	$\Rightarrow$ s_0 0.00	$\Rightarrow$ 0.00	$\Uparrow$ 0.00	$\Leftarrow$ 0.00

$\hat{V}_\pi^0$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Iterative Policy Evaluation for SSPs: Example

	1	2	3	4
5	$\Rightarrow$ 1.00	$\Rightarrow$ 1.00	$\Rightarrow$ 1.00	$s_\star$ 0.00
4	$\Rightarrow$ 1.00	$\Uparrow$ 1.00	$\Uparrow$ 3.00	$\Uparrow$ 1.00
3	$\Rightarrow$ 1.00	$\Uparrow$ 1.00	$\Leftarrow$ 1.00	$\Leftarrow$ 1.00
2	$\Uparrow$ 1.00	$\Uparrow$ 1.00	$\Uparrow$ 1.00	$\Leftarrow$ 1.00
1	$\Rightarrow$ s_0 1.00	$\Rightarrow$ 1.00	$\Uparrow$ 1.00	$\Leftarrow$ 1.00

$\hat{V}_\pi^1$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Iterative Policy Evaluation for SSPs: Example

	1	2	3	4
5	$\Rightarrow$ 2.00	$\Rightarrow$ 2.00	$\Rightarrow$ 1.00	$s_\star$ 0.00
4	$\Rightarrow$ 2.00	$\Uparrow$ 2.00	$\Uparrow$ 5.20	$\Uparrow$ 1.60
3	$\Rightarrow$ 2.00	$\Uparrow$ 2.00	$\Leftarrow$ 2.00	$\Leftarrow$ 2.00
2	$\Uparrow$ 2.00	$\Uparrow$ 2.00	$\Uparrow$ 2.00	$\Leftarrow$ 2.00
1	$\Rightarrow$ s_0 2.00	$\Rightarrow$ 2.00	$\Uparrow$ 2.00	$\Leftarrow$ 2.00

$\hat{V}_\pi^2$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Iterative Policy Evaluation for SSPs: Example

	1	2	3	4
5	⇒ 3.96	⇒ 2.00	⇒ 1.00	$s_\star$ 0.00
4	⇒ 4.60	⇑ 3.00	⇑ 7.79	⇑ 2.31
3	⇒ 5.00	⇑ 4.00	⇐ 5.00	⇐ 5.00
2	⇑ 5.00	⇑ 5.00	⇑ 5.00	⇐ 5.00
1	⇒ s_0 5.00	⇒ 5.00	⇑ 5.00	⇐ 5.00

$\hat{V}_\pi^5$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Iterative Policy Evaluation for SSPs: Example

	1	2	3	4
5	⇒ 4.46	⇒ 2.00	⇒ 1.00	$s_\star$ 0.00
4	⇒ 5.43	⇑ 3.00	⇑ 8.44	⇑ 2.50
3	⇒ 6.38	⇑ 4.00	⇐ 5.00	⇐ 7.31
2	⇑ 8.30	⇑ 6.38	⇑ 6.00	⇐ 8.18
1	⇒ s_0 9.00	⇒ 8.00	⇑ 7.00	⇐ 8.96

$\hat{V}_\pi^{10}$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Iterative Policy Evaluation for SSPs: Example

	1	2	3	4
5	⇒ 4.50	⇒ 2.00	⇒ 1.00	$s_\star$ 0.00
4	⇒ 5.50	⇑ 3.00	⇑ 8.50	⇑ 2.50
3	⇒ 6.50	⇑ 4.00	⇐ 5.00	⇐ 7.50
2	⇑ 9.00	⇑ 6.50	⇑ 6.00	⇐ 8.50
1	⇒ s_0 9.00	⇒ 8.00	⇑ 7.00	⇐ 9.50

$\hat{V}_\pi^{29}$

- cost of 3 to move from striped cells (cost is 1 otherwise)
- moving from gray cells unsuccessful with probability 0.6

Iterative Policy Evaluation: Algorithm

Iterative Policy Evaluation for SSP $\langle S, A, c, T, s_0, S_\star \rangle$, complete policy π and $\epsilon > 0$

initialize $\hat{V}^0$ to 0 for goal states, otherwise arbitrarily
for $i = 1, 2, \ldots$:
  for all states $s \in S \setminus S_\star$:
    $\hat{V}_\pi^i(s) := c(\pi(s)) + \sum_{s' \in S} T(s, \pi(s), s') \cdot \hat{V}_\pi^{i-1}(s')$
  if $\max_{s \in S} |\hat{V}_\pi^i(s) - \hat{V}_\pi^{i-1}(s)| < \epsilon$:
    return $\hat{V}_\pi^i$

Iterative Policy Evaluation: Properties

Theorem (Convergence of Iterative Policy Evaluation)

Let $\mathcal{T} = \langle S, A, c, T, s_0, S_\star \rangle$ be an SSP, π be a proper policy for $\mathcal{T}$ and $\hat{V}^0_\pi(s) \in \mathbb{R}$ arbitrarily for all $s \setminus S_\star$.

Iterative policy evaluation converges to the true state-values, i.e.,

$$\lim_{i \to \infty} \hat{V}^i_\pi(s) = V_\pi(s) \text{ for all } s \in S.$$

Proof omitted.

In practice, iterative policy evaluation converges to true state-values if ϵ is small enough.

Policy Evaluation: MDPs

What about policy evaluation for MDPs?

- MDPs (with finite state set) are always cyclic
  $\Rightarrow$ backward induction not applicable
- but goal state not required for iterative policy evaluation
- albeit traces are infinite, iterative policy evaluation converges
- convergence theorem also holds for MDPs

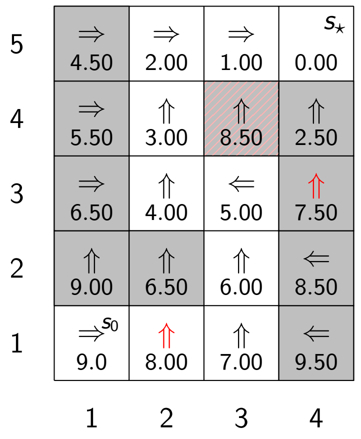
F3.3 Policy Improvement

Example: Greedy Action

	1	2	3	4
5	⇒ 4.50	⇒ 2.00	⇒ 1.00	$s_\star$ 0.00
4	⇒ 5.50	⇑ 3.00	⇑ 8.50	⇑ 2.50
3	⇒ 6.50	⇑ 4.00	⇐ 5.00	⇐ 7.50
2	⇑ 9.00	⇑ 6.50	⇑ 6.00	⇐ 8.50
1	⇒ s_0 9.0	⇒ 8.00	⇑ 7.00	⇐ 9.50

- Can we learn more from this than the state-values of a policy?

Example: Greedy Action



- Can we learn more from this than the state-values of a policy?
- Yes! By evaluating all actions in each state, we can derive a better policy

Greedy Actions and Policies for SSPs

Definition (Greedy Action)

Let s be a state of an SSP $\mathcal{T} = \langle S, A, c, T, s_0, S_\star \rangle$ and V be a state-value function for $\mathcal{T}$.
The set of greedy actions in s with respect to V is

$$A_V(s) := \arg\min_{a \in A(s)} \left(c(a) + \sum_{s' \in S} T(s, a, s') \cdot V(s') \right).$$

A policy π_V with $\pi_V(s) \in A_V(s)$ is a greedy policy.

Determining a greedy policy of a given state-value function is called policy improvement.

Greedy Actions and Policies for MDPs

Definition (Greedy Action)

Let s be a state of a (discounted-reward) MDP $\mathcal{T} = \langle S, A, R, T, s_0, \gamma \rangle$ and V be a state-value function for $\mathcal{T}$.
The set of greedy actions in s with respect to V is

$$A_V(s) := \arg\max_{a \in A(s)} \left(R(s, a) + \gamma \sum_{s' \in S} T(s, a, s') \cdot V(s') \right).$$

A policy π_V with $\pi_V(s) \in A_V(s)$ is a greedy policy.

Determining a greedy policy of a given state-value function is called policy improvement.

F3.4 Policy Iteration

Policy Iteration

- Policy Iteration (PI) was first proposed by Howard in 1960
- based on the observation that the greedy actions describe a better policy
- starts with arbitrary policy π_0
- alternates policy evaluation and policy improvement
- as long as policy changes

Example: Policy Iteration

	1	2	3	4
5	⇒ 4.50	⇒ 2.00	⇒ 1.00	$s_\star$ 0.00
4	⇒ 5.50	⇑ 3.00	⇑ 8.50	⇑ 2.50
3	⇒ 6.50	⇑ 4.00	⇐ 5.00	⇐ 7.50
2	⇑ 9.00	⇑ 6.50	⇑ 6.00	⇐ 8.50
1	⇒ s_0 9.00	⇒ 8.00	⇑ 7.00	⇐ 9.50

π_0

Example: Policy Iteration

	1	2	3	4
5	⇒ 4.50	⇒ 2.00	⇒ 1.00	$s_\star$ 0.00
4	⇒ 5.50	⇑ 3.00	⇑ 8.50	⇑ 2.50
3	⇒ 6.50	⇑ 4.00	⇐ 5.00	⇑ 5.00
2	⇑ 9.00	⇑ 6.50	⇑ 6.00	⇐ 8.50
1	⇒ s_0 8.50	⇑ 7.50	⇑ 7.00	⇐ 9.50

π_1

Example: Policy Iteration

	1	2	3	4
5	⇒ 4.50	⇒ 2.00	⇒ 1.00	$s_\star$ 0.00
4	⇒ 5.50	⇑ 3.00	⇑ 8.50	⇑ 2.50
3	⇒ 6.50	⇑ 4.00	⇐ 5.00	⇑ 5.00
2	⇑ 9.00	⇑ 6.50	⇑ 6.00	⇑ 7.50
1	⇒ s_0 8.50	⇑ 7.50	⇑ 7.00	⇐ 9.50

$\pi_2 = \pi_3$

Policy Iteration: Algorithm

Policy Iteration for SSP or MDP $\mathcal{T}$

initialize π_0 to any policy (for SSP: proper)
for $i = 0, 1, \dots$:
  compute V_{π_i}
  let π_{i+1} be a greedy policy w.r.t V_{π_i}
  if $\pi_i = \pi_{i+1}$:
    return π_i

Note: if $\pi_i(s) \in A_{V_{\pi_i}(s)}$ then use $\pi_{i+1}(s) := \pi_i(s)$
(only update the policy where necessary).

Properties

- PI computes optimal policy if policy evaluation is exact
- In practice, PI often requires very few iterations ...
- ... and is much faster than solving an LP

F3.5 Summary

Summary

- Policy evaluation for an acyclic policy is possible in one sweep over the state space with backward induction
- Iterative policy evaluation applies state-value function iteratively and converges to true state-values
- Greedy actions in evaluated policy allow to improve policy
- Policy iteration alternates policy evaluation and policy improvement
- Policy iteration computes an optimal policy (if policy evaluation is exact)