Planning and Optimization

F2. Bellman Equation & Linear Programming

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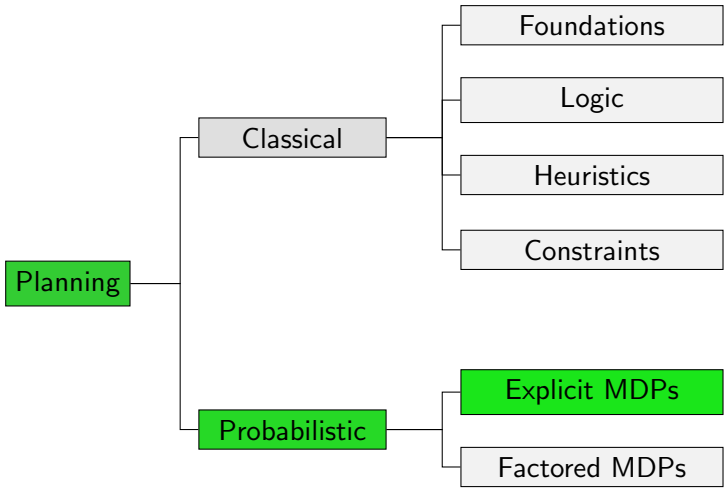
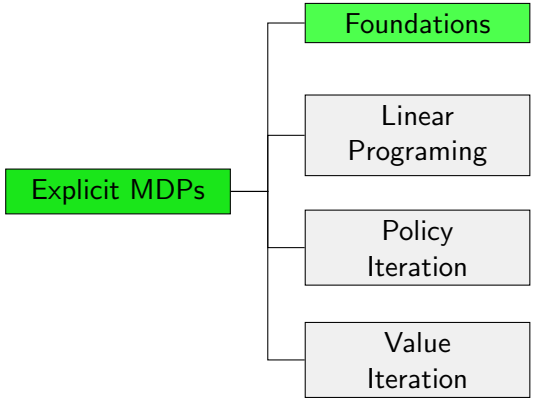
Content of this Course



Content of this Course: Explicit MDPs



F2.1 Introduction

Quality of Solutions

- Solution in classical planning: plan
- Optimality criterion of a solution in classical planning: minimize plan cost
- Solution in probabilistic planning: policy
- What is the optimality criterion of a solution in probabilistic planning?

Example: Swiss Lotto

Example (Swiss Lotto)

What is the expected payoff of placing one bet in Swiss Lotto for a cost of CHF 2.50 with (simplified) payouts and probabilities:

CHF 30.000.000 with prob. $1/31474716$	$(6+1)$
CHF 1.000.000 with prob. $1/5245786$	(6)
CHF 5.000 with prob. $1/850668$	(5)
CHF 50 with prob. $1/111930$	(4)
CHF 10 with prob. $1/11480$	(3)

Solution: $\frac{30000000}{31474716} + \frac{1000000}{5245786} + \frac{5000}{850668} + \frac{50}{111930} + \frac{10}{11480} - 2.5 \approx -1.35.$

Expected Values under Uncertainty

Definition (Expected Value of a Random Variable)

Let X be a random variable with a finite number of outcomes $d_1, \ldots, d_n \in \mathbb{R}$, and let d_i happen with probability $p_i \in [0, 1]$ (for $i = 1, \ldots n$) s.t. $\sum_{i=1}^{n} p_i = 1$.

The expected value of X is $\mathbb{E}[X] = \sum_{i=1}^{n} (p_i \cdot d_i)$.

F2.2 Bellman Equation

Value Functions for MDPs

Definition (Value Functions for MDPs)

Let π be a policy for MDP $\mathcal{T} = \langle S, A, R, T, s_0, \gamma \rangle$.

The state-value $V_\pi(s)$ of $s \in S_\pi(s_0)$ under π is defined as

$$V_\pi(s) := Q_\pi(s, \pi(s))$$

where the action-value $Q_\pi(s, a)$ of s and a under π is defined as

$$Q_\pi(s, a) := R(s, a) + \gamma \cdot \sum_{s' \in \text{succ}(s,a)} T(s, a, s') \cdot V_\pi(s').$$

The state-value $V_\pi(s)$ describes the expected reward of applying π in MDP $\mathcal{T}$, starting from s.

Bellman Equation in MDPs

Definition (Bellman Equation in MDPs)

Let $\mathcal{T} = \langle S, A, R, T, s_0, \gamma \rangle$ be an MDP.

The Bellman equation for a state s of $\mathcal{T}$ is the set of equations that describes $V_\star(s)$, where

$$V_\star(s) := \max_{a \in A(s)} Q_\star(s, a)$$

$$Q_\star(s, a) := R(s, a) + \gamma \cdot \sum_{s' \in \text{succ}(s,a)} T(s, a, s') \cdot V_\star(s').$$

The solution $V_\star(s)$ of the Bellman equation describes the maximal expected reward that can be achieved from state s in MDP $\mathcal{T}$.

Optimal Policy in MDPs

What is the policy that achieves the maximal expected reward?

Definition (Optimal Policy in MDPs)

Let $\mathcal{T} = \langle S, A, R, T, s_0, \gamma \rangle$ be an MDP.

A policy π is an optimal policy if $\pi(s) \in \arg\max_{a \in A(s)} Q_\star(s, a)$ for all $s \in S_\pi(s_0)$ and the expected reward of π in $\mathcal{T}$ is $V_\star(s_0)$.

Value Functions for SSPs

Definition (Value Functions for SSPs)

Let $\mathcal{T} = \langle S, A, c, T, s_0, S_\star \rangle$ be an SSP and π be a policy for $\mathcal{T}$.

The state-value $V_\pi(s)$ of s under π is defined as

$$V_\pi(s) := \begin{cases} 0 & \text{if } s \in S_\star \\ Q_\pi(s, \pi(s)) & \text{otherwise,} \end{cases}$$

where the action-value $Q_\pi(s, a)$ of s and a under π is defined as

$$Q_\pi(s, a) := c(a) + \sum_{s' \in \mathrm{succ}(s,a)} T(s, a, s') \cdot V_\pi(s').$$

The state-value $V_\pi(s)$ describes the expected cost of applying π in SSP $\mathcal{T}$, starting from s.

Bellman Equation in SSPs

Definition (Bellman Equation in SSPs)

Let $\mathcal{T} = \langle S, A, c, T, s_0, S_\star \rangle$ be an SSP.

The Bellman equation for a state s of $\mathcal{T}$ is the set of equations that describes $V_\star(s)$, where

$$V_\star(s) := \begin{cases} 0 & \text{if } s \in S_\star \\ \min_{a \in A(s)} Q_\star(s, a) & \text{otherwise,} \end{cases}$$

$$Q_\star(s, a) := c(a) + \sum_{s' \in \mathrm{succ}(s,a)} T(s, a, s') \cdot V_\star(s').$$

The solution $V_\star(s)$ of the Bellman equation describes the minimal expected cost that can be achieved from state s in SSP $\mathcal{T}$.

Optimal Policy in SSPs

What is the policy that achieves the minimal expected cost?

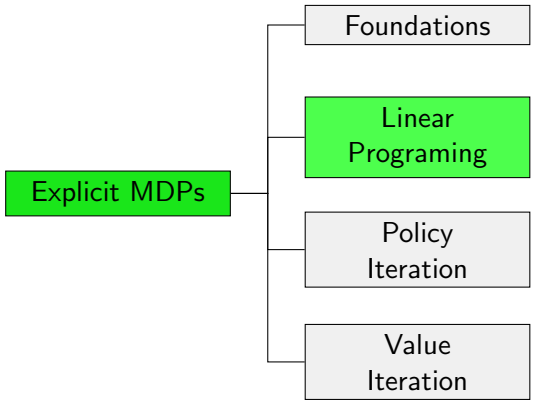
Definition (Optimal Policy in SSPs)

Let $\mathcal{T} = \langle S, A, c, T, s_0, S_\star \rangle$ be an SSP.

A policy π is an optimal policy if $\pi(s) \in \arg\min_{a \in A(s)} Q_\star(s, a)$ for all $s \in S_\pi(s_0) \setminus S_\star$ and the expected cost of π in $\mathcal{T}$ is $V_\star(s_0)$.

F2.3 Linear Programming

Content of this Course: Explicit MDPs



Linear Programming for SSPs

- Bellman equation: set of equations that describes the expected cost for each state.
- there are $|S|$ variables and $|S|$ equations (replacing $Q_\star$ in $V_\star$ with the corresponding equation)
- If we solve these equations, we can determine an optimal policy for the SSP from the state-values.
- Problem: how can we deal with the minimization?

$\Rightarrow$ We have solved the "same" problem before with the help of an LP solver

Reminder: LP for Shortest Path in State Space

Variables
Non-negative variable Distance_s for each state s

Objective
Maximize Distance_{s_0}

Subject to

$$\text{Distance}_{s_\star} = 0 \qquad \text{for all goal states } s_\star$$

$$\text{Distance}_s \leq \text{Distance}_{s'} + c(\ell) \text{ for all transitions } s \xrightarrow{\ell} s'$$

LP for Expected Cost in SSP

Variables
Non-negative variable ExpCost_s for each state s

Objective
Maximize ExpCost_{s_0}

Subject to

$$\text{ExpCost}_{s_\star} = 0 \qquad \text{for all goal states } s_\star$$

$$\text{ExpCost}_s \leq \Big(\sum_{s' \in S} T(s, a, s') \cdot \text{ExpCost}_{s'}\Big) + c(a)$$

$$\text{for all } s \in S \text{ and } a \in A(s)$$

LP for Expected Reward in MDP

Variables

Non-negative variable ExpReward_s for each state s

Objective

Minimize ExpReward_{s_0}

Subject to

$$\text{ExpReward}_s \geq (\gamma \cdot \sum_{s' \in S} T(s, a, s') \text{ExpReward}_{s'}) + R(s, a)$$

for all $s \in S$ and $A \in A(s)$

Complexity of Probabilistic Planning

- an optimal solution for MDPs or SSPs can be computed with an LP solver
- requires $|S|$ variables and $|S| \cdot |A|$ constraints
- we know that LPs can be solved in polynomial time
- $\Rightarrow$ solving MDPs or SSPs is a polynomial time problem

How does this relate to the complexity result for classical planning?

Solving MDPs or SSPs is polynomial in $|S| \cdot |A|$.

F2.4 Summary

Summary

- The state-values of a policy specify the expected reward (cost) of following that policy.
- The Bellman equation describes the state-values of an optimal policy.
- Linear Programming can be used to solve MDPs and SSPs in time polynomial in the size of the MDP/SSP.