

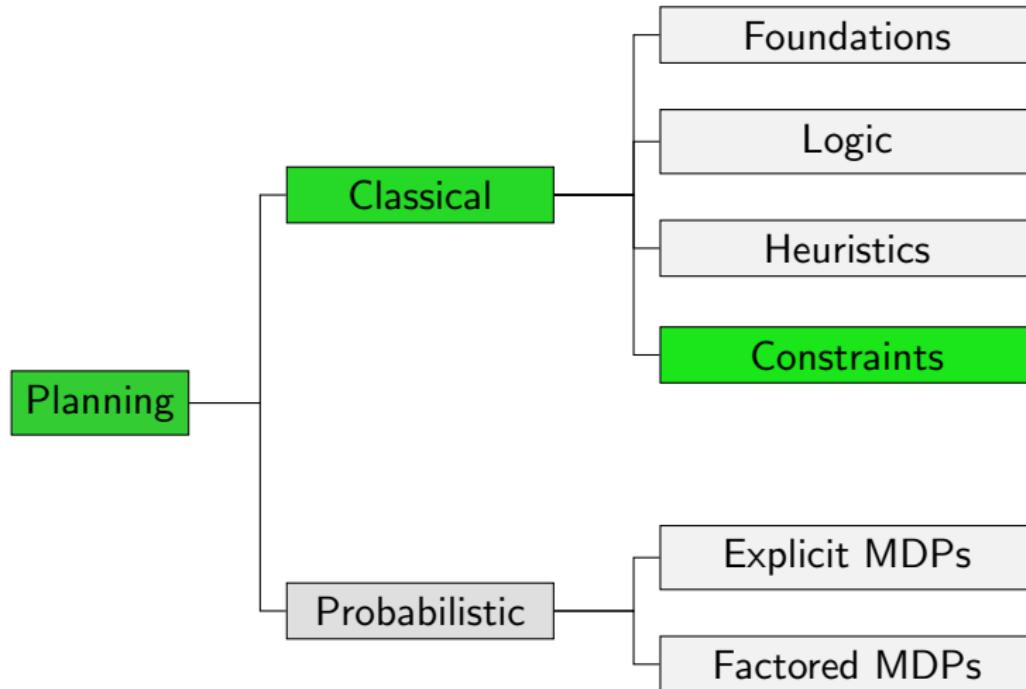
# Planning and Optimization

## E4. Linear & Integer Programming

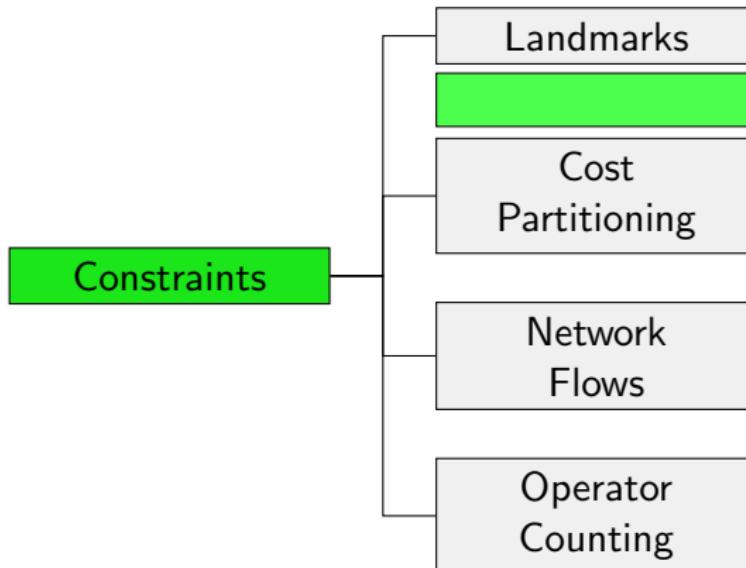
Malte Helmert and Gabriele Röger

Universität Basel

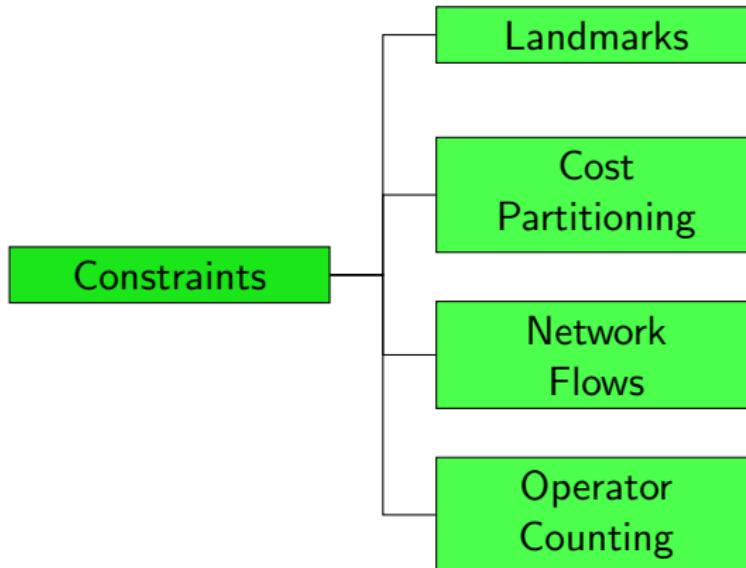
# Content of this Course



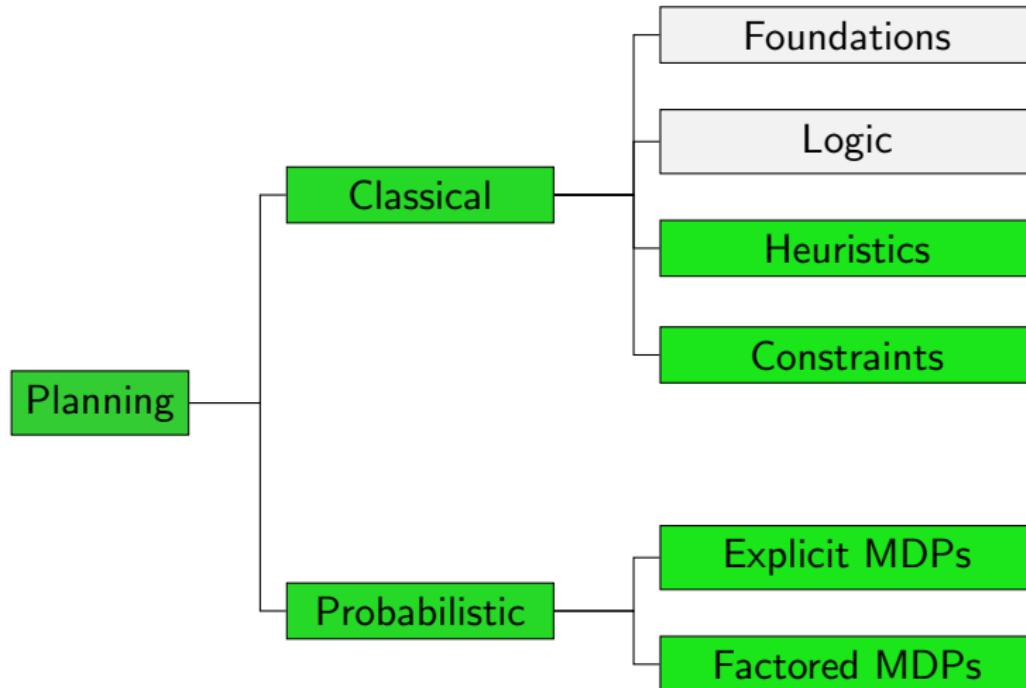
# Content of this Course: Constraints (Timeline)



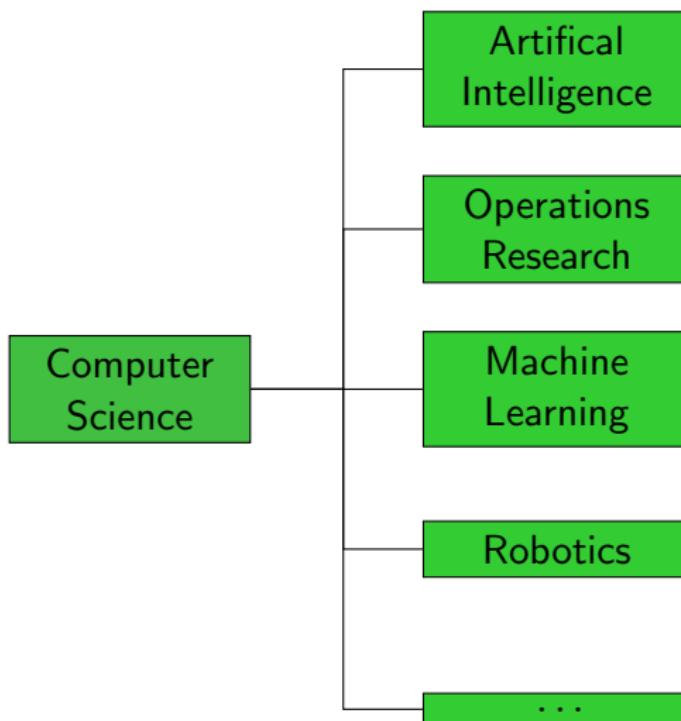
# Content of this Course: Constraints (Relevance)



# Content of this Course (Relevance)



# Content of this Course (Relevance)



# Integer Programs

# Motivation

- This goes on beyond Computer Science
- Active **research** on IPs and LPs in
  - Operation Research
  - Mathematics
- Many **application** areas, for instance:
  - Manufacturing
  - Agriculture
  - Mining
  - Logistics
  - Planning
- As an application, we treat LPs / IPs as a **blackbox**
- We just look at **the fundamentals**

# Motivation

## Example (Optimization Problem)

Consider the following scenario:

- A factory produces two products A and B
- Selling a unit of B yields 5 times the profit of a unit of A.
- A client places the unusual order to “buy anything that can be produced on that day as long as the units of B do not exceed two plus twice the units of A.”
- The factory can produce at most 12 products per day.
- There is only material for 6 units of A  
(there is enough material to produce any amount of B)

How many units of A and B does the client receive  
if the factory owner aims to maximize her profit?

# Integer Program: Example

Let  $X_A$  and  $X_B$  be the (integer) number of produced A and B

## Example (Optimization Problem as Integer Program)

$$X_A \geq 0, \quad X_B \geq 0$$

## Example (Optimization Problem)

## Integer Program: Example

Let  $X_A$  and  $X_B$  be the (integer) number of produced A and B

### Example (Optimization Problem as Integer Program)

maximize  $X_A + 5X_B$  subject to

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### Example (Optimization Problem)

- “a unit of B yields 5 times the profit of a unit of A”
- “the factory owner aims to maximize her profit”

## Integer Program: Example

Let  $X_A$  and  $X_B$  be the (integer) number of produced A and B

### Example (Optimization Problem as Integer Program)

maximize  $X_A + 5X_B$  subject to

$$2 + 2X_A \geq X_B$$

$$X_A \geq 0, \quad X_B \geq 0$$

### Example (Optimization Problem)

- “the units of B may not exceed two plus twice the units of A.”

## Integer Program: Example

Let  $X_A$  and  $X_B$  be the (integer) number of produced A and B

### Example (Optimization Problem as Integer Program)

maximize  $X_A + 5X_B$  subject to

$$2 + 2X_A \geq X_B$$

$$X_A + X_B \leq 12$$

$$X_A \geq 0, \quad X_B \geq 0$$

### Example (Optimization Problem)

- “The factory can produce at most 12 units per day”

## Integer Program: Example

Let  $X_A$  and  $X_B$  be the (integer) number of produced A and B

### Example (Optimization Problem as Integer Program)

maximize  $X_A + 5X_B$  subject to

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$$X_A + X_B \leq 12$$

$$X_A \leq 6$$

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### Example (Optimization Problem)

- “There is only material for 6 units of A”

# Integer Program: Example

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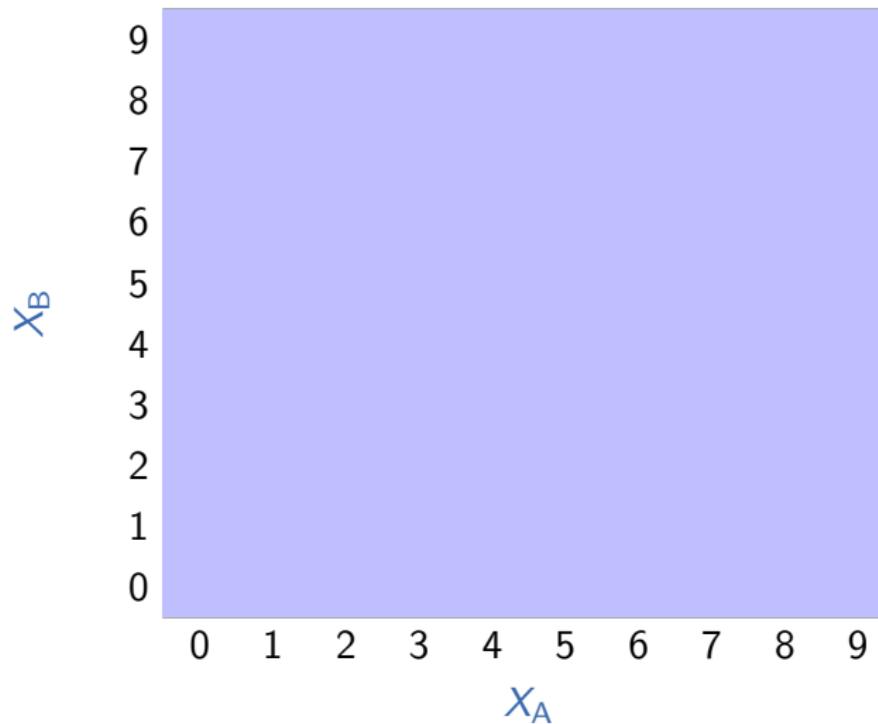
$$X_A \leq 6$$

$$X_A \geq 0, \quad X_B \geq 0$$

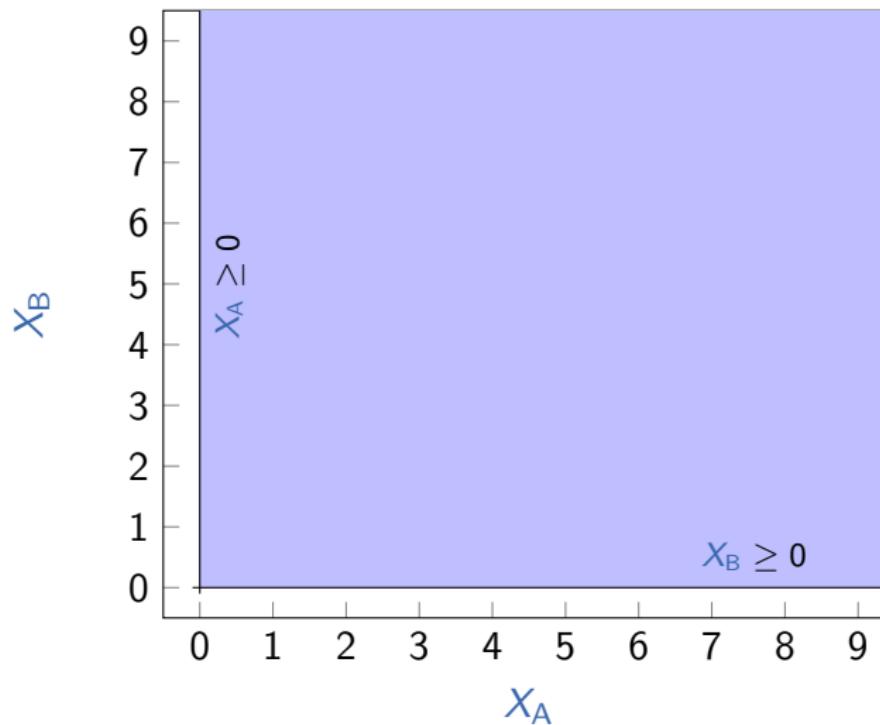
~ unique optimal solution:

produce 4 A ( $X_A = 4$ ) and 8 B ( $X_B = 8$ ) for a profit of 44

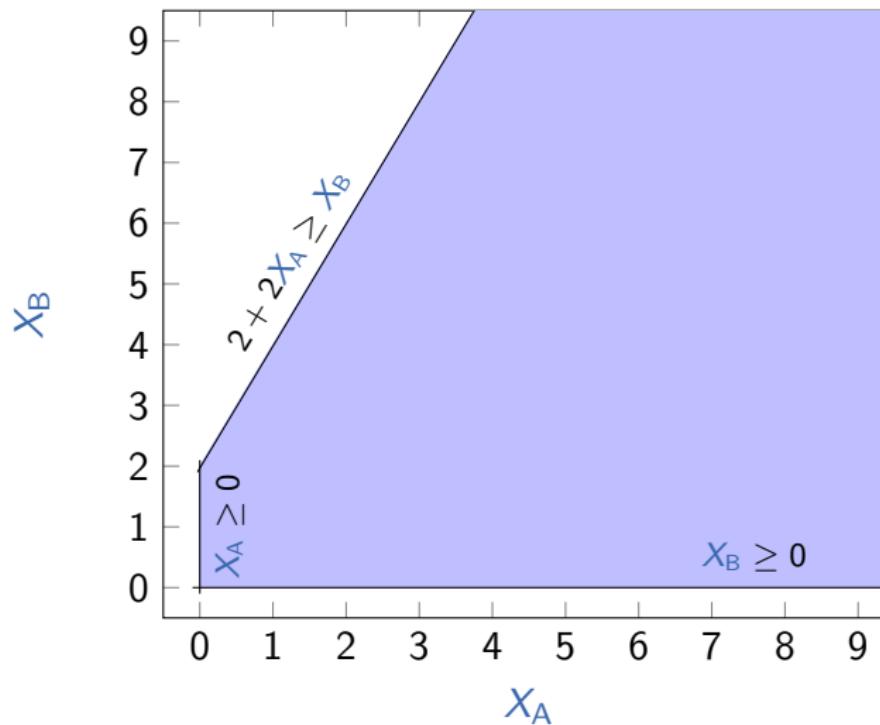
# Integer Program Example: Visualization



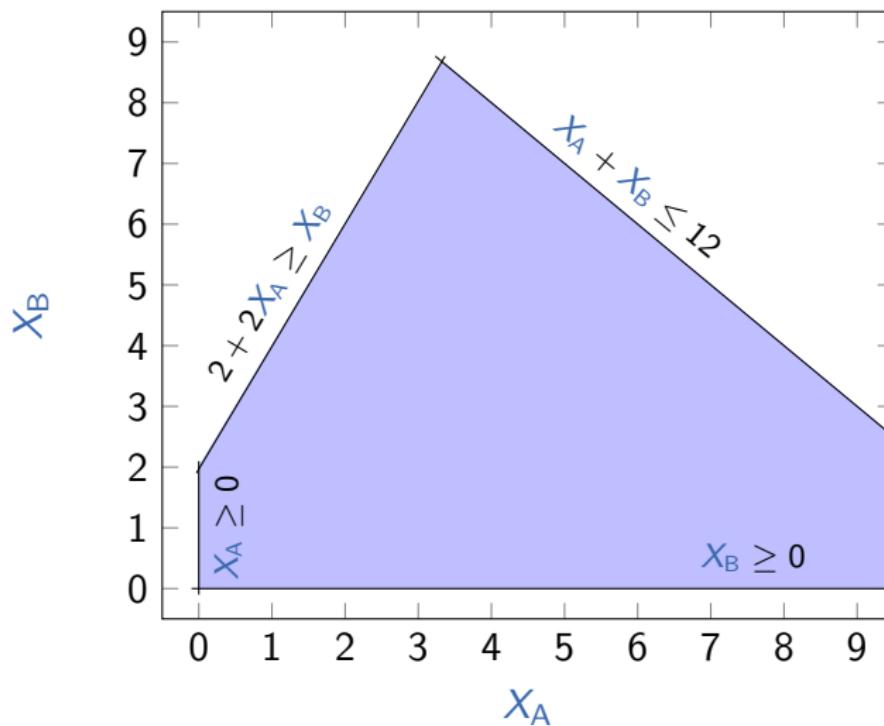
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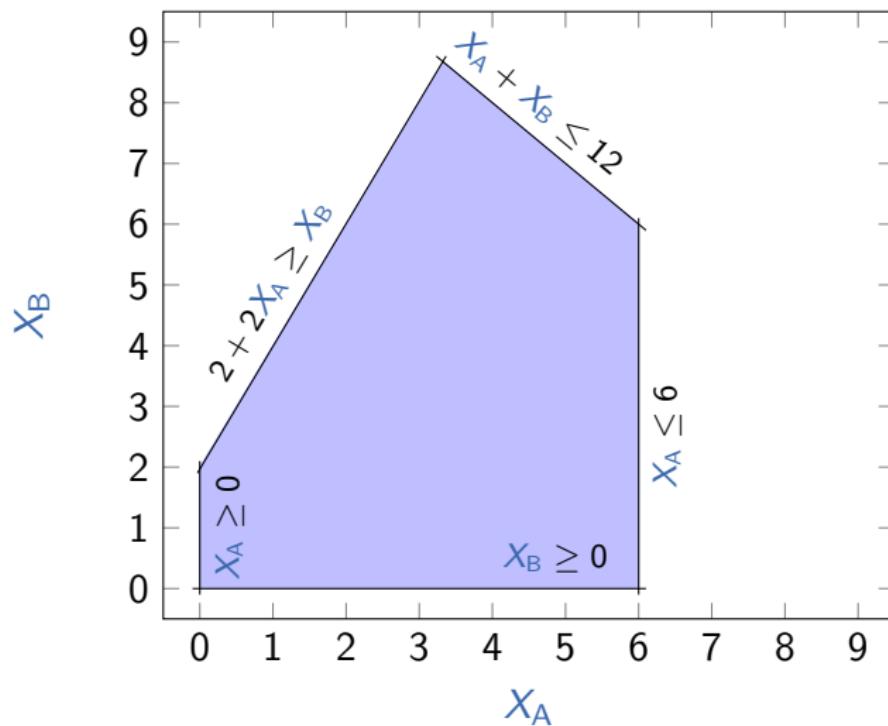
## Integer Program Example: Visualization



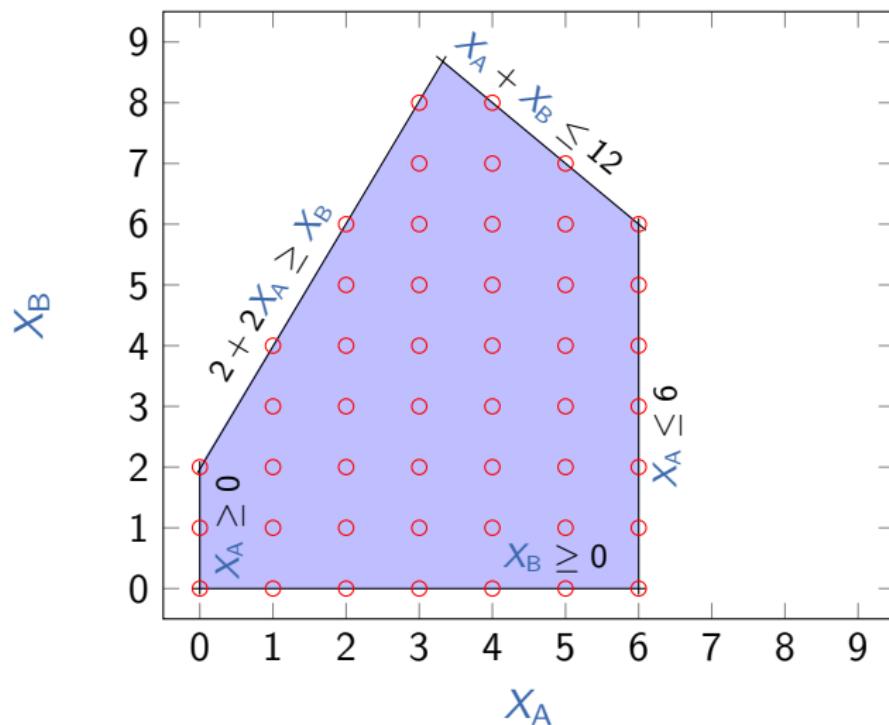
## Integer Program Example: Visualization



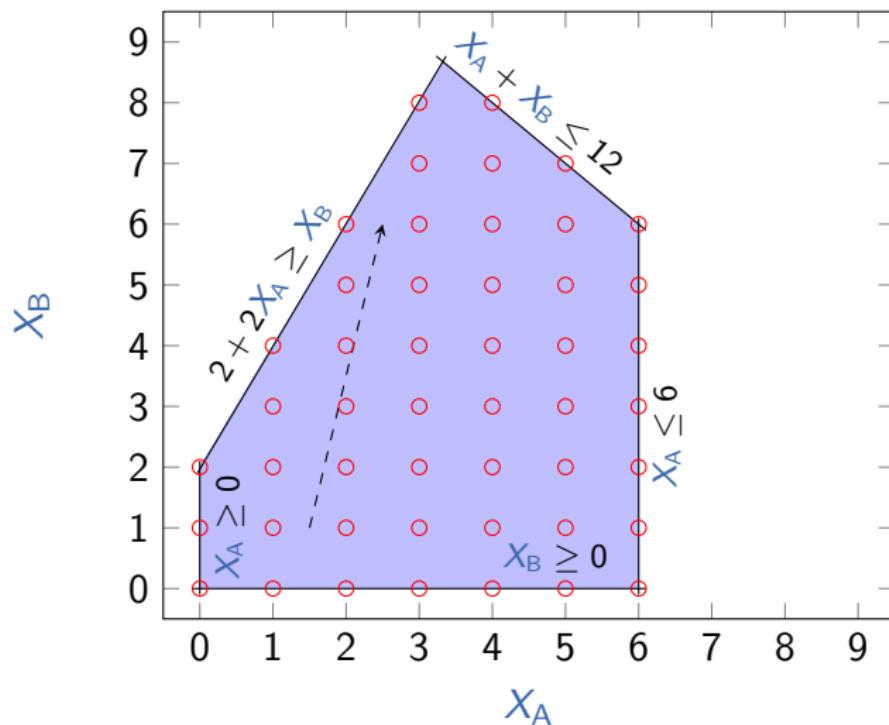
## Integer Program Example: Visualization



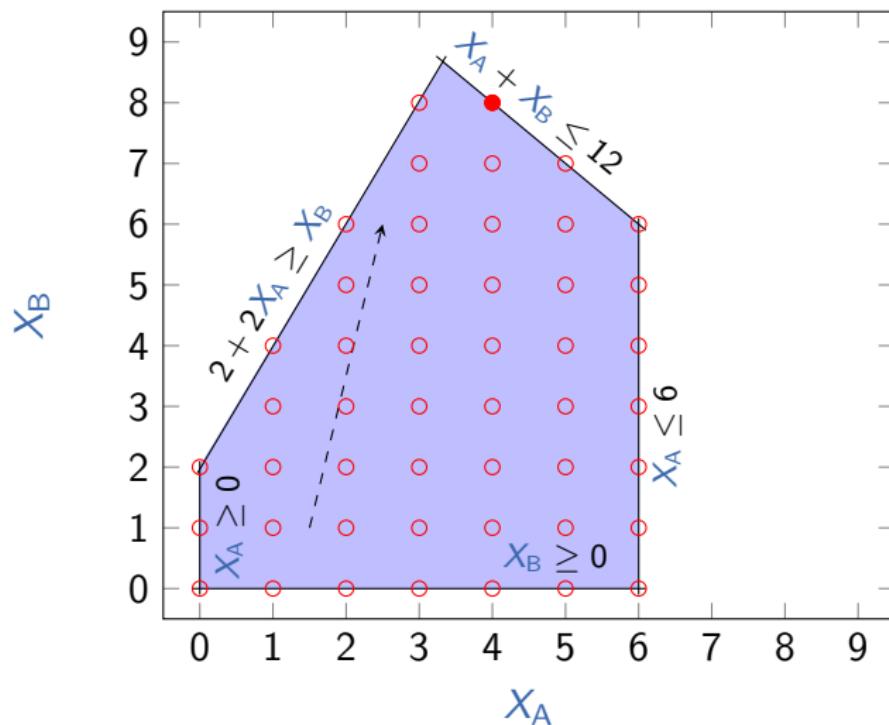
## Integer Program Example: Visualization



## Integer Program Example: Visualization



## Integer Program Example: Visualization



# Integer Programs

## Integer Program

An **integer program (IP)** consists of:

- a finite set of **integer-valued variables**  $V$
- a finite set of **linear inequalities** (constraints) over  $V$
- an **objective function**, which is a linear combination of  $V$
- which should be **minimized** or **maximized**.

# Terminology

- An integer assignment to all variables in  $V$  is **feasible** if it satisfies the constraints.
- An integer program is **feasible** if there is such a feasible assignment. Otherwise it is **infeasible**.
- A feasible maximum (resp. minimum) problem is **unbounded** if the objective function can assume arbitrarily large positive (resp. negative) values at feasible assignments. Otherwise it is **bounded**.
- The **objective value** of a bounded feasible maximum (resp. minimum) problem is the maximum (resp. minimum) value of the objective function with a feasible assignment.

## Three Classes of IPs

IPs fall into three classes:

- **bounded feasible:** IP is solvable and optimal solutions exist
- **unbounded feasible:** IP is solvable and arbitrarily good solutions exist
- **infeasible:** IP is unsolvable

# Another Example

## Example

minimize  $3X_{o_1} + 4X_{o_2} + 5X_{o_3}$  subject to

$$X_{o_4} \geq 1$$

$$X_{o_1} + X_{o_2} \geq 1$$

$$X_{o_1} + X_{o_3} \geq 1$$

$$X_{o_2} + X_{o_3} \geq 1$$

$$X_{o_1} \geq 0, \quad X_{o_2} \geq 0, \quad X_{o_3} \geq 0, \quad X_{o_4} \geq 0$$

What example from a previous chapter does this IP encode?

# Another Example

## Example

minimize  $3X_{o_1} + 4X_{o_2} + 5X_{o_3}$  subject to

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$$X_{o_1} + X_{o_3} \geq 1$$

$$X_{o_2} + X_{o_3} \geq 1$$

$$X_{o_1} \geq 0, \quad X_{o_2} \geq 0, \quad X_{o_3} \geq 0, \quad X_{o_4} \geq 0$$

What example from a previous chapter does this IP encode?

~ the minimum hitting set from Chapter E2

# Complexity of Solving Integer Programs

- As an IP can compute an MHS, solving an IP must be at least as complex as computing an MHS
  - Reminder: MHS is a “classical” NP-complete problem
  - Good news: Solving an IP is not harder
- ~~> Finding solutions for IPs is **NP-complete**.

# Complexity of Solving Integer Programs

- As an IP can compute an MHS, solving an IP must be at least as complex as computing an MHS
  - Reminder: MHS is a “classical” NP-complete problem
  - Good news: Solving an IP is not harder
- ~~> Finding solutions for IPs is **NP-complete**.

Removing the requirement that solutions must be integer-valued leads to a simpler problem

# Linear Programs

# Linear Programs

## Linear Program

A **linear program (LP)** consists of:

- a finite set of **real-valued variables**  $V$
- a finite set of **linear inequalities** (constraints) over  $V$
- an **objective function**, which is a linear combination of  $V$
- which should be **minimized** or **maximized**.

We use the introduced IP terminology also for LPs.

**Mixed IPs (MIPs)** generalize IPs and LPs:  
some variables are integer-values, some are real-valued.

# Linear Program: Example

Let  $X_A$  and  $X_B$  be the (real-valued) number of produced A and B

## Example (Optimization Problem as Linear Program)

maximize  $X_A + 5X_B$  subject to

$$2 + 2X_A \geq X_B$$

$$X_A + X_B \leq 12$$

$$X_A \leq 6$$

$$X_A \geq 0, \quad X_B \geq 0$$

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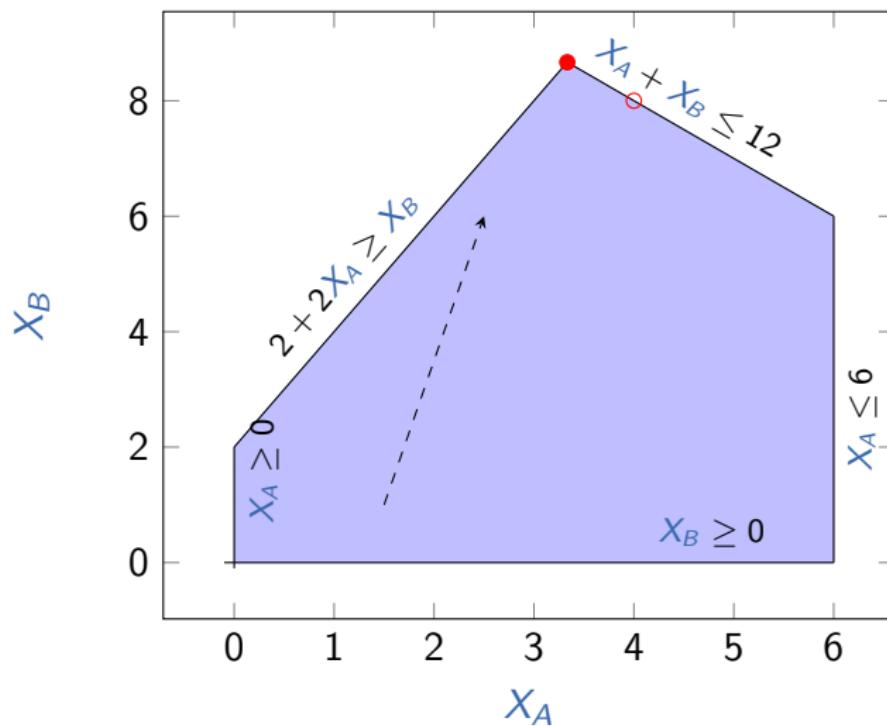
$$X_A \leq 6$$

$$X_A \geq 0, \quad X_B \geq 0$$

⇒ unique optimal solution:

$X_A = 3\frac{1}{3}$  and  $X_B = 8\frac{2}{3}$  with objective value  $46\frac{2}{3}$

## Linear Program Example: Visualization



# Solving Linear Programs

## ■ Observation:

For an maximization problem, the objective value of the LP is  
**not lower** than the one of the IP.

# Solving Linear Programs

- **Observation:**

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- **Complexity:**

LP solving is a **polynomial-time** problem.

# Solving Linear Programs

- **Observation:**

For an maximization problem, the objective value of the LP is **not lower** than the one of the IP.

- **Complexity:**

LP solving is a **polynomial-time** problem.

- **Common idea:**

Approximate IP objective value with corresponding LP  
**(LP relaxation).**

# LP Relaxation

## Theorem (LP Relaxation)

*The LP relaxation of an integer program is the problem that arises by removing the requirement that variables are integer-valued.*

*For a maximization (resp. minimization) problem, the objective value of the LP relaxation is an upper (resp. lower) bound on the value of the IP.*

## Proof idea.

Every feasible assignment for the IP is also feasible for the LP.

## LP Relaxation of MHS heuristic

## Example (Minimum Hitting Set)

minimize  $3X_{o_1} + 4X_{o_2} + 5X_{o_3}$  subject to

$$X_{o_4} \geq 1$$

$$X_{o_1} + X_{o_2} \geq 1$$

$$X_{o_1} + X_{o_3} \geq 1$$

$$X_{o_2} + X_{o_3} \geq 1$$

$$X_{o_1} \geq 0, \quad X_{o_2} \geq 0, \quad X_{o_3} \geq 0, \quad X_{o_4} \geq 0$$

~~ optimal solution of LP relaxation:

$X_{o_4} = 1$  and  $X_{o_1} = X_{o_2} = X_{o_3} = 0.5$  with objective value 6

~~ LP relaxation of MHS heuristic is admissible  
and can be computed in polynomial time

## Some LP Theory: Duality

Every LP has an alternative view (its **dual** LP).

- roughly: variables and constraints swap roles
- roughly: objective coefficients and bounds swap roles
- dual of maximization LP is minimization LP and vice versa
- dual of dual: original LP

# Duality Theorem

## Theorem (Duality Theorem)

*If a linear program is **bounded feasible**, then so is its dual, and their **objective values are equal**.*

(Proof omitted.)

The dual provides a different perspective on a problem.

# Summary

# Summary

- Linear (and integer) programs consist of an **objective function** that should be **maximized or minimized** subject to a set of given **linear constraints**.
- Finding solutions for **integer** programs is **NP-complete**.
- **LP solving** is a **polynomial time** problem.
- The dual of a maximization LP is a minimization LP and vice versa.
- The **dual** of a bounded feasible LP has the **same objective value**.

## Further Reading

The slides in this chapter are based on the following excellent tutorial on LP solving:



Thomas S. Ferguson.  
Linear Programming – A Concise Introduction.  
UCLA, unpublished document available online.