Planning and Optimization

B8. Symbolic Search: Full Algorithm

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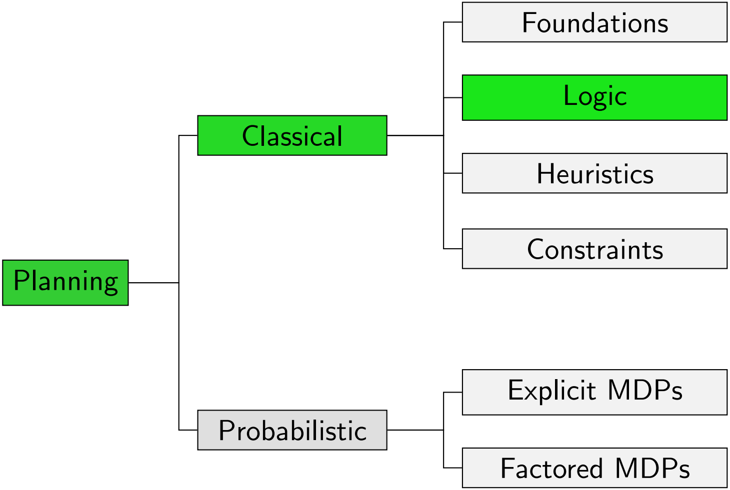
Content of this Course



Planning

Classical

Probabilistic

Foundations

Logic

Heuristics

Constraints

Explicit MDPs

Factored MDPs

Devising a Symbolic Search Algorithm

- We now put the pieces together to build a symbolic search algorithm for propositional planning tasks.
- use BDDs as a black box data structure:
    - care about provided operations and their time complexity
    - do not care about their internal implementation
- Efficient implementations are available as libraries, e.g.:
    - CUDD, a high-performance BDD library
    - libbdd, shipped with Ubuntu Linux

Basic BDD Operations

BDD Operations: Preliminaries

- All BDDs work on a fixed and totally ordered set of propositional variables.
- Complexity of operations given in terms of:
    - k, the number of BDD variables
    - $\|B\|$, the number of nodes in the BDD B

BDD Operations (1)

BDD operations: logical/set atoms

- bdd-true(): build BDD representing all assignments
    - in logic: $\top$
    - time complexity: $O(1)$
- bdd-false(): build BDD representing $\emptyset$
    - in logic: $\bot$
    - time complexity: $O(1)$
- bdd-atom(v): build BDD representing $\{s \mid s(v) = 1\}$
    - in logic: v
    - time complexity: $O(1)$

BDD Operations (2)

BDD operations: logical/set connectives

- bdd-complement(B): build BDD representing $\overline{r(B)}$
    - in logic: $\neg\varphi$
    - time complexity: $O(\|B\|)$
- bdd-union(B, B'): build BDD representing $r(B) \cup r(B')$
    - in logic: $(\varphi \vee \psi)$
    - time complexity: $O(\|B\| \cdot \|B'\|)$
- analogously:
    - bdd-intersection(B, B'): $r(B) \cap r(B')$, $(\varphi \wedge \psi)$
    - bdd-setdifference(B, B'): $r(B) \setminus r(B')$, $(\varphi \wedge \neg\psi)$
    - bdd-implies(B, B'): $\overline{r(B)} \cup r(B')$, $(\varphi \rightarrow \psi)$
    - bdd-equiv(B, B'): $(r(B) \cap r(B')) \cup (\overline{r(B)} \cap \overline{r(B')})$, $(\varphi \leftrightarrow \psi)$

BDD Operations (3)

BDD operations: Boolean tests

- bdd-includes(B, I): return **true** iff $I \in r(B)$
    - in logic: $I \models \varphi$?
    - time complexity: $O(k)$
- bdd-equals(B, B'): return **true** iff $r(B) = r(B')$
    - in logic: $\varphi \equiv \psi$?
    - time complexity: $O(1)$ (due to canonical representation)

Conditioning: Formulas

The last two basic BDD operations are a bit more unusual and require some preliminary remarks.

Conditioning a variable v in a formula φ to $\mathbf{T}$ or $\mathbf{F}$, written $\varphi[\mathbf{T}/v]$ or $\varphi[\mathbf{F}/v]$, means restricting v to a particular truth value:

Examples:

- $(A \land (B \lor \neg C))[\mathbf{T}/B] = (A \land (\top \lor \neg C)) \equiv A$
- $(A \land (B \lor \neg C))[\mathbf{F}/B] = (A \land (\bot \lor \neg C)) \equiv A \land \neg C$

Conditioning: Sets of Assignments

We can define the same operation for sets of assignments S: $S[\mathbf{F}/v]$ and $S[\mathbf{T}/v]$ restrict S to elements with the given value for v and remove v from the domain of definition:

Example:

- $S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}, C \mapsto \mathbf{F}\}, \{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{F}\}, \{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$

$\rightsquigarrow$ $S[\mathbf{T}/B] = \{\{A \mapsto \mathbf{T}, C \mapsto \mathbf{F}\}, \{A \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$

Forgetting

Forgetting (a.k.a. existential abstraction) is similar to conditioning: we allow either truth value for v and remove the variable.

We write this as $\exists v\, \varphi$ (for formulas) and $\exists v\, S$ (for sets).

Formally:

- $\exists v\, \varphi = \varphi[\mathbf{T}/v] \vee \varphi[\mathbf{F}/v]$
- $\exists v\, S = S[\mathbf{T}/v] \cup S[\mathbf{F}/v]$

Forgetting: Example

Examples:

- $S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}, C \mapsto \mathbf{F}\},$
  $\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{F}\},$
  $\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$

$\rightsquigarrow$ $\exists B\, S = \{\{A \mapsto \mathbf{F}, C \mapsto \mathbf{F}\},$
$\{A \mapsto \mathbf{T}, C \mapsto \mathbf{F}\},$
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$\rightsquigarrow$ $\exists C\, S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}\},$
$\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}\}\}$

BDD Operations (4)

BDD operations: conditioning and forgetting

- bdd-condition(B, v, t) where $t \in \{\mathbf{T}, \mathbf{F}\}$:
  build BDD representing $r(B)[t/v]$
    - in logic: $\varphi[t/v]$
    - time complexity: $O(\|B\|)$
- bdd-forget(B, v):
  build BDD representing $\exists v\, r(B)$
    - in logic: $\exists v\, \varphi \quad (= \varphi[\mathbf{T}/v] \lor \varphi[\mathbf{F}/v])$
    - time complexity: $O(\|B\|^2)$

Formulas and Singletons

Formulas to BDDs

- With the logical/set operations, we can convert propositional formulas φ into BDDs representing the models of φ.
- We denote this computation with bdd-formula(φ).
- Each individual logical connective takes polynomial time, but converting a full formula of length n can take $O(2^n)$ time. (How is this possible?)

Singleton BDDs

- We can convert a single truth assignment I into a BDD representing $\{I\}$ by computing the conjunction of all literals true in I (using bdd-atom, bdd-complement and bdd-intersection).
- We denote this computation with bdd-singleton(I).
- When done in the correct order, this takes time $O(k)$.

Renaming

Renaming

We will need to support one final operation on formulas: renaming.

Renaming X to Y in formula φ, written $\varphi[X \to Y]$, means replacing all occurrences of X by Y in φ.

We require that Y is not present in φ initially.

Example:

- $\varphi = (A \land (B \lor \neg C))$

$\rightsquigarrow$ $\varphi[A \to D] = (D \land (B \lor \neg C))$

How Hard Can That Be?

- For formulas, renaming is a simple (linear-time) operation.
- For a BDD B, it is equally simple ($O(\|B\|)$) when renaming between variables that are adjacent in the variable order.
- In general, it requires $O(\|B\|^2)$, using the equivalence $\varphi[X \to Y] \equiv \exists X(\varphi \wedge (X \leftrightarrow Y))$

Symbolic Breadth-first Search

Planning Task State Variables vs. BDD Variables

Consider propositional planning task $\langle V, I, O, \gamma \rangle$ with states S.

In symbolic planning, we have two BDD variables v and v' for every state variable $v \in V$ of the planning task.

- use unprimed variables v to describe sets of states: $\{s \in S \mid \text{some property}\}$
- use combinations of unprimed and primed variables v, v' to describe sets of state pairs: $\{\langle s, s' \rangle \mid \text{some property}\}$

Breadth-first Search with Progression and BDDs

Progression Breadth-first Search

```
def bfs-progression($V$, $I$, $O$, $\gamma$):
    $goal\_states := models(\gamma)$
    $reached_0 := \{I\}$
    $i := 0$
    loop:
        if $reached_i \cap goal\_states \neq \emptyset$:
            return solution found
        $reached_{i+1} := reached_i \cup apply(reached_i, O)$
        if $reached_{i+1} = reached_i$:
            return no solution exists
        $i := i + 1$
```

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Use *bdd-formula*.

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Use *bdd-singleton*.

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Use *bdd-intersection*, *bdd-false*, *bdd-equals*.

Breadth-first Search with Progression and BDDs

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        if $reached_{i+1} = reached_i$:
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```

Use *bdd-union*.

Breadth-first Search with Progression and BDDs

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```

Use *bdd-equals*.

Breadth-first Search with Progression and BDDs

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How to do this?

The *apply* Function (1)

We need an operation that

- for a set of states *reached* (given as a BDD)
- and a set of operators O
- computes the set of states (as a BDD) that can be reached by applying some operator $o \in O$ in some state $s \in \mathit{reached}$.

We have seen something similar already. . .

Translating Operators into Formulas

Definition (Operators in Propositional Logic)

Let o be an operator and V a set of state variables.

Define $\tau_V(o) := \mathit{pre}(o) \land \bigwedge_{v \in V}(\mathit{regr}(v, \mathit{eff}(o)) \leftrightarrow v')$.

States that o is applicable and describes how

- the new value of v, represented by v',
- must relate to the old state, described by variables V.

The *apply* Function (2)

- The formula $\tau_V(o)$ describes all transitions $s \xrightarrow{o} s'$
    - induced by a single operator o
    - in terms of variables V describing s
    - and variables V' describing s'.
- The formula $\bigvee_{o \in O} \tau_V(o)$ describes state transitions by any operator in O.
- We can translate this formula to a BDD (over variables $V \cup V'$) with *bdd-formula*.
- The resulting BDD is called the transition relation of the planning task, written as $T_V(O)$.

The *apply* Function (3)

Using the transition relation, we can compute *apply*(*reached*, *O*) as follows:

The apply function

```
def apply(reached, O):
    $B := T_V(O)$
    $B :=$ bdd-intersection$(B, reached)$
    for each $v \in V$:
        $B :=$ bdd-forget$(B, v)$
    for each $v \in V$:
        $B :=$ bdd-rename$(B, v', v)$
    return $B$
```

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        $B := bdd\text{-}rename(B, v', v)$
    return $B$
```

This describes the set of state pairs $\langle s, s' \rangle$ where s' is a successor of s in terms of variables $V \cup V'$.

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    $B :=$ bdd-intersection($B$, $reached$)
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    for each $v \in V$:
        $B :=$ bdd-rename($B$, $v'$, $v$)
    return $B$
```

This describes the set of state pairs $\langle s, s' \rangle$ where s' is a successor of s and $s \in$ *reached* in terms of variables $V \cup V'$.

The *apply* Function (3)

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The apply function

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def apply($reached$, $O$):
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This describes the set of states s' which are successors of some state $s \in reached$ in terms of variables V'.

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This describes the set of states s' which are successors of some state $s \in reached$ in terms of variables V.

The *apply* Function (3)

Using the transition relation, we can compute *apply*$(reached, O)$ as follows:

The apply function

```
def apply($reached$, $O$):
    $B := T_V(O)$
    $B :=$ $bdd\text{-}intersection(B, reached)$
    for each $v \in V$:
        $B :=$ $bdd\text{-}forget(B, v)$
    for each $v \in V$:
        $B :=$ $bdd\text{-}rename(B, v', v)$
    return $B$
```

Thus, *apply* indeed computes the set of successors of *reached* using operators O.

Discussion

Discussion

- This completes the discussion of a (basic) symbolic search algorithm for classical planning.
- We ignored the aspect of solution extraction.
  This needs some extra work, but is not a major challenge.
- In practice, some steps can be performed slightly more efficiently, but these are comparatively minor details.

Variable Orders

For good performance, we need a good variable ordering.

- Variables that refer to the same state variable before and after operator application (v and v') should be neighbors in the transition relation BDD.

Finite-Domain Variables and Variable Orders

The algorithm can easily be extended to FDR tasks by using $\lceil \log_2 n \rceil$ BDD variables to represent a state variable with n possible values.

- Variables related to the same FDR variable should be kept together in the BDD variable ordering (but still interleaving primed and unprimed variables).
- Automatic conversion from STRIPS to SAS$^+$ was first explored in the context of symbolic search.
- It was found critical for performance.

Extensions

Symbolic search can be extended to. . .

- regression and bidirectional search:
  this is very easy and often effective
- uniform-cost search:
  requires some work, but not too difficult in principle
- heuristic search:
  requires a heuristic representable as a BDD;
  has not really been shown to outperform blind symbolic search

Literature

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Summary

Summary

- Symbolic search operates on sets of states instead of individual states as in explicit-state search.
- State sets and transition relations can be represented as BDDs.
- Based on this, we can implement a blind breadth-first search in an efficient way.
- A good variable ordering is crucial for performance.