

# Discrete Mathematics in Computer Science

## E5. Syntax and Semantics of Predicate Logic

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## — E5. Syntax and Semantics of Predicate Logic

### E5.1 Syntax of Predicate Logic

### E5.2 Semantics of Predicate Logic

## E5.1 Syntax of Predicate Logic

## Limits of Propositional Logic

Cannot well be expressed in propositional logic:

- ▶ “Everyone who does the exercises passes the exam.”
- ▶ “If someone with administrator privileges presses ‘delete’, all data is gone.”
- ▶ “Everyone has a mother.”
- ▶ “If someone is the father of some person, the person is his child.”

▷ need more expressive logic

↪ **predicate logic** (a.k.a. first-order logic)

German: Prädikatenlogik (erster Stufe)

## Syntax: Building Blocks

- ▶ **Signatures** define allowed symbols.  
analogy: atom set  $A$  in propositional logic
- ▶ **Terms** are associated with objects by the semantics.  
no analogy in propositional logic
- ▶ **Formulas** are associated with truth values (**true** or **false**) by the semantics.  
analogy: formulas in propositional logic

German: Signatur, Term, Formel

## Signatures: Definition

### Definition (Signature)

A **signature** (of predicate logic) is a 4-tuple  $S = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  consisting of the following four disjoint sets:

- ▶ a finite or countable set  $\mathcal{V}$  of **variable symbols**
- ▶ a finite or countable set  $\mathcal{C}$  of **constant symbols**
- ▶ a finite or countable set  $\mathcal{F}$  of **function symbols**
- ▶ a finite or countable set  $\mathcal{P}$  of **predicate symbols** (or **relation symbols**)

Every function symbol  $f \in \mathcal{F}$  and predicate symbol  $P \in \mathcal{P}$  has an associated **arity**  $ar(f), ar(P) \in \mathbb{N}_1$  (number of arguments).

German: Variablen-, Konstanten-, Funktions-, Prädikat- und Relationssymbole; Stelligkeit

## Signatures: Terminology and Conventions

terminology:

- ▶  **$k$ -ary** (function or predicate) symbol: symbol  $s$  with arity  $ar(s) = k$ .
- ▶ also: **unary**, **binary**, **ternary**

German:  $k$ -stellig, unär, binär, ternär

conventions (in this course):

- ▶ variable symbols written in *italics*, other symbols upright.
- ▶ predicate symbols begin with capital letter, other symbols with lower-case letters

## Signatures: Examples

### Example: Arithmetic

- ▶  $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- ▶  $\mathcal{C} = \{\text{zero}, \text{one}\}$
- ▶  $\mathcal{F} = \{\text{sum}, \text{product}\}$
- ▶  $\mathcal{P} = \{\text{Positive}, \text{SquareNumber}\}$

$ar(\text{sum}) = ar(\text{product}) = 2$ ,  $ar(\text{Positive}) = ar(\text{SquareNumber}) = 1$

## Signatures: Examples

### Example: Genealogy

- ▶  $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- ▶  $\mathcal{C} = \{\text{roger-federer, lisa-simpson}\}$
- ▶  $\mathcal{F} = \emptyset$
- ▶  $\mathcal{P} = \{\text{Female, Male, Parent}\}$

$ar(\text{Female}) = ar(\text{Male}) = 1, ar(\text{Parent}) = 2$

## Terms: Definition

### Definition (Term)

Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature.

A **term** (over  $\mathcal{S}$ ) is inductively constructed according to the following rules:

- ▶ Every variable symbol  $v \in \mathcal{V}$  is a term.
- ▶ Every constant symbol  $c \in \mathcal{C}$  is a term.
- ▶ If  $t_1, \dots, t_k$  are terms and  $f \in \mathcal{F}$  is a function symbol with arity  $k$ , then  $f(t_1, \dots, t_k)$  is a term.

German: Term

examples:

- ▶  $x_4$
- ▶ lisa-simpson
- ▶  $\text{sum}(x_3, \text{product}(\text{one}, x_5))$

## Formulas: Definition

### Definition (Formula)

For a signature  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  the set of predicate logic formulas (over  $\mathcal{S}$ ) is inductively defined as follows:

- ▶ If  $t_1, \dots, t_k$  are terms (over  $\mathcal{S}$ ) and  $P \in \mathcal{P}$  is a  $k$ -ary predicate symbol, then the **atomic formula** (or the **atom**)  $P(t_1, \dots, t_k)$  is a formula over  $\mathcal{S}$ .
- ▶ If  $t_1$  and  $t_2$  are terms (over  $\mathcal{S}$ ), then the **identity**  $(t_1 = t_2)$  is a formula over  $\mathcal{S}$ .
- ▶ If  $x \in \mathcal{V}$  is a variable symbol and  $\varphi$  a formula over  $\mathcal{S}$ , then the **universal quantification**  $\forall x \varphi$  and the **existential quantification**  $\exists x \varphi$  are formulas over  $\mathcal{S}$ .

...

German: atomare Formel, Atom, Identität, Allquantifizierung, Existenzquantifizierung

## Formulas: Definition

### Definition (Formula)

For a signature  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  the set of predicate logic formulas (over  $\mathcal{S}$ ) is inductively defined as follows:

...

- ▶ If  $\varphi$  is a formula over  $\mathcal{S}$ , then so is its **negation**  $\neg\varphi$ .
- ▶ If  $\varphi$  and  $\psi$  are formulas over  $\mathcal{S}$ , then so are the **conjunction**  $(\varphi \wedge \psi)$  and the **disjunction**  $(\varphi \vee \psi)$ .

German: Negation, Konjunktion, Disjunktion

## Formulas: Examples

### Examples: Arithmetic and Genealogy

- ▶  $\text{Positive}(x_2)$
- ▶  $\forall x (\neg \text{SquareNumber}(x) \vee \text{Positive}(x))$
- ▶  $\exists x_3 (\text{SquareNumber}(x_3) \wedge \neg \text{Positive}(x_3))$
- ▶  $\forall x (x = y)$
- ▶  $\forall x (\text{sum}(x, x) = \text{product}(x, \text{one}))$
- ▶  $\forall x \exists y (\text{sum}(x, y) = \text{zero})$
- ▶  $\forall x \exists y (\text{Parent}(y, x) \wedge \text{Female}(y))$

**Terminology:** The symbols  $\forall$  and  $\exists$  are called **quantifiers**.

**German:** Quantoren

## Abbreviations and Placement of Parentheses by Convention

### abbreviations:

- ▶  $(\varphi \rightarrow \psi)$  is an abbreviation for  $(\neg\varphi \vee \psi)$ .
- ▶  $(\varphi \leftrightarrow \psi)$  is an abbreviation for  $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$ .
- ▶ Sequences of the same quantifier can be abbreviated.  
For example:
  - ▶  $\forall x \forall y \forall z \varphi \rightsquigarrow \forall xyz \varphi$
  - ▶  $\exists x \exists y \exists z \varphi \rightsquigarrow \exists xyz \varphi$
  - ▶  $\forall w \exists x \exists y \forall z \varphi \rightsquigarrow \forall w \exists xy \forall z \varphi$

### placement of parentheses by convention:

- ▶ analogous to propositional logic
- ▶ quantifiers  $\forall$  and  $\exists$  bind more strongly than anything else.
- ▶ example:  $\forall x P(x) \rightarrow Q(x)$  corresponds to  $(\forall x P(x) \rightarrow Q(x))$ ,  
not  $\forall x (P(x) \rightarrow Q(x))$ .

## Exercise

$\mathcal{S} = \langle \{x, y, z\}, \{c\}, \{f, g, h\}, \{Q, R, S\} \rangle$  with  
 $ar(f) = 3, ar(g) = ar(h) = 1, ar(Q) = 2, ar(R) = ar(S) = 1$

- ▶  $f(x, y)$
- ▶  $(g(x) = R(y))$
- ▶  $(g(x) = f(y, c, h(x)))$
- ▶  $(R(x) \wedge \forall x S(x))$
- ▶  $\forall c Q(c, x)$
- ▶  $(\forall x \exists y (g(x) = y) \vee (h(x) = c))$

Which expressions are syntactically correct formulas or terms for  $\mathcal{S}$ ?  
 What kind of term/formula?

## E5.2 Semantics of Predicate Logic

## Semantics: Motivation

- ▶ interpretations in propositional logic: truth assignments for the **propositional variables**
- ▶ There are no propositional variables in predicate logic.
- ▶ instead: interpretation determines meaning of the **constant, function and predicate symbols**.
- ▶ meaning of **variable symbols** not determined by interpretation but by separate **variable assignment**

## Interpretations and Variable Assignments

Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature.

### Definition (Interpretation, Variable Assignment)

An **interpretation** (for  $\mathcal{S}$ ) is a pair  $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  of:

- ▶ a non-empty set  $U$  called the **universe** and
- ▶ a function  $\cdot^{\mathcal{I}}$  that assigns a meaning to the constant, function, and predicate symbols:
  - ▶  $c^{\mathcal{I}} \in U$  for constant symbols  $c \in \mathcal{C}$
  - ▶  $f^{\mathcal{I}} : U^k \rightarrow U$  for  $k$ -ary function symbols  $f \in \mathcal{F}$
  - ▶  $P^{\mathcal{I}} \subseteq U^k$  for  $k$ -ary predicate symbols  $P \in \mathcal{P}$

A **variable assignment** (for  $\mathcal{S}$  and universe  $U$ ) is a function  $\alpha : \mathcal{V} \rightarrow U$ .

**German:** Interpretation, Universum (or Grundmenge), Variablenzuweisung

## Interpretations and Variable Assignments: Example

### Example

signature:  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  with  $\mathcal{V} = \{x, y, z\}$ ,  
 $\mathcal{C} = \{\text{zero}, \text{one}\}$ ,  $\mathcal{F} = \{\text{sum}, \text{product}\}$ ,  $\mathcal{P} = \{\text{SquareNumber}\}$   
 $ar(\text{sum}) = ar(\text{product}) = 2$ ,  $ar(\text{SquareNumber}) = 1$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  with

- ▶  $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- ▶  $\text{zero}^{\mathcal{I}} = u_0$
- ▶  $\text{one}^{\mathcal{I}} = u_1$
- ▶  $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$  for all  $i, j \in \{0, \dots, 6\}$
- ▶  $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$  for all  $i, j \in \{0, \dots, 6\}$
- ▶  $\text{SquareNumber}^{\mathcal{I}} = \{u_0, u_1, u_2, u_4\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

## Semantics: Informally

**Example:**  $(\forall x(\text{Block}(x) \rightarrow \text{Red}(x)) \wedge \text{Block}(a))$   
 “For all objects  $x$ : if  $x$  is a block, then  $x$  is red.  
 Also, the object called  $a$  is a block.”

- ▶ **Terms** are interpreted as **objects**.
- ▶ **Unary predicates** denote properties of objects (to be a block, to be red, to be a square number, ...).
- ▶ **General predicates** denote relations between objects (to be someone’s child, to have a common divisor, ...).
- ▶ **Universally quantified** formulas (“ $\forall$ ”) are true if they hold for **every** object in the universe.
- ▶ **Existentially quantified** formulas (“ $\exists$ ”) are true if they hold for **at least one** object in the universe.

## Interpretations of Terms

Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature.

### Definition (Interpretation of a Term)

Let  $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  be an interpretation for  $\mathcal{S}$ , and let  $\alpha$  be a variable assignment for  $\mathcal{S}$  and universe  $U$ .

Let  $t$  be a term over  $\mathcal{S}$ .

The **interpretation of  $t$**  under  $\mathcal{I}$  and  $\alpha$ , written as  $t^{\mathcal{I}, \alpha}$ , is the element of the universe  $U$  defined as follows:

- ▶ If  $t = x$  with  $x \in \mathcal{V}$  ( $t$  is a **variable term**):  
 $x^{\mathcal{I}, \alpha} = \alpha(x)$
- ▶ If  $t = c$  with  $c \in \mathcal{C}$  ( $t$  is a **constant term**):  
 $c^{\mathcal{I}, \alpha} = c^{\mathcal{I}}$
- ▶ If  $t = f(t_1, \dots, t_k)$  ( $t$  is a **function term**):  
 $f(t_1, \dots, t_k)^{\mathcal{I}, \alpha} = f^{\mathcal{I}}(t_1^{\mathcal{I}, \alpha}, \dots, t_k^{\mathcal{I}, \alpha})$

## Interpretations of Terms: Example

### Example

signature:  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$

with  $\mathcal{V} = \{x, y, z\}$ ,  $\mathcal{C} = \{\text{zero}, \text{one}\}$ ,  $\mathcal{F} = \{\text{sum}, \text{product}\}$ ,

$ar(\text{sum}) = ar(\text{product}) = 2$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  with

- ▶  $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- ▶  $\text{zero}^{\mathcal{I}} = u_0$
- ▶  $\text{one}^{\mathcal{I}} = u_1$
- ▶  $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$  for all  $i, j \in \{0, \dots, 6\}$
- ▶  $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$  for all  $i, j \in \{0, \dots, 6\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

## Interpretations of Terms: Example (ctd.)

### Example (ctd.)

- ▶  $\text{zero}^{\mathcal{I}, \alpha} =$
- ▶  $y^{\mathcal{I}, \alpha} =$
- ▶  $\text{sum}(x, y)^{\mathcal{I}, \alpha} =$
- ▶  $\text{product}(\text{one}, \text{sum}(x, \text{zero}))^{\mathcal{I}, \alpha} =$

## Semantics of Predicate Logic Formulas

Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature.

### Definition (Formula is Satisfied or True)

Let  $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  be an interpretation for  $\mathcal{S}$ , and let  $\alpha$  be a variable assignment for  $\mathcal{S}$  and universe  $U$ .

We say that  $\mathcal{I}$  and  $\alpha$  **satisfy** a predicate logic formula  $\varphi$  (also:  $\varphi$  is **true** under  $\mathcal{I}$  and  $\alpha$ ), written:  $\mathcal{I}, \alpha \models \varphi$ , according to the following inductive rules:

$$\begin{aligned} \mathcal{I}, \alpha \models P(t_1, \dots, t_k) & \text{ iff } \langle t_1^{\mathcal{I}, \alpha}, \dots, t_k^{\mathcal{I}, \alpha} \rangle \in P^{\mathcal{I}} \\ \mathcal{I}, \alpha \models (t_1 = t_2) & \text{ iff } t_1^{\mathcal{I}, \alpha} = t_2^{\mathcal{I}, \alpha} \\ \mathcal{I}, \alpha \models \neg \varphi & \text{ iff } \mathcal{I}, \alpha \not\models \varphi \\ \mathcal{I}, \alpha \models (\varphi \wedge \psi) & \text{ iff } \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi \\ \mathcal{I}, \alpha \models (\varphi \vee \psi) & \text{ iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \quad \dots \end{aligned}$$

**German:**  $\mathcal{I}$  und  $\alpha$  erfüllen  $\varphi$  (also:  $\varphi$  ist wahr unter  $\mathcal{I}$  und  $\alpha$ )

## Semantics of Predicate Logic Formulas

Let  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$  be a signature.

### Definition (Formula is Satisfied or True)

...

$\mathcal{I}, \alpha \models \forall x \varphi$  iff  $\mathcal{I}, \alpha[x := u] \models \varphi$  for all  $u \in U$

$\mathcal{I}, \alpha \models \exists x \varphi$  iff  $\mathcal{I}, \alpha[x := u] \models \varphi$  for at least one  $u \in U$

where  $\alpha[x := u]$  is the same variable assignment as  $\alpha$ , except that it maps variable  $x$  to the value  $u$ .

Formally:

$$(\alpha[x := u])(z) = \begin{cases} u & \text{if } z = x \\ \alpha(z) & \text{if } z \neq x \end{cases}$$

## Semantics: Example

### Example

signature:  $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$

with  $\mathcal{V} = \{x, y, z\}$ ,  $\mathcal{C} = \{a, b\}$ ,  $\mathcal{F} = \emptyset$ ,  $\mathcal{P} = \{\text{Block}, \text{Red}\}$ ,

$ar(\text{Block}) = ar(\text{Red}) = 1$ .

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$  with

▶  $U = \{u_1, u_2, u_3, u_4, u_5\}$

▶  $a^{\mathcal{I}} = u_1$

▶  $b^{\mathcal{I}} = u_3$

▶  $\text{Block}^{\mathcal{I}} = \{u_1, u_2\}$

▶  $\text{Red}^{\mathcal{I}} = \{u_1, u_2, u_3, u_5\}$

$\alpha = \{x \mapsto u_1, y \mapsto u_2, z \mapsto u_1\}$

## Semantics: Example (ctd.)

### Example (ctd.)

Questions:

▶  $\mathcal{I}, \alpha \models (\text{Block}(b) \vee \neg \text{Block}(b))?$

▶  $\mathcal{I}, \alpha \models (\text{Block}(x) \rightarrow (\text{Block}(x) \vee \neg \text{Block}(y)))?$

▶  $\mathcal{I}, \alpha \models (\text{Block}(a) \wedge \text{Block}(b))?$

▶  $\mathcal{I}, \alpha \models \forall x (\text{Block}(x) \rightarrow \text{Red}(x))?$